Realization of effective super Tonks-Girardeau gases via strongly attractive one-dimensional Fermi gases

Shu Chen,[1] Xi-Wen Guan,[2] Xiangguo Yin,[1] Liming Guan,[1] and M. T. Batchelor[2,3]

[1]*Institute of Physics, Chinese Academy of Sciences, Beijing 100190, People's Republic of China*

[2]*Department of Theoretical Physics, Research School of Physics and Engineering, Australian National University, Canberra ACT 0200, Australia*

[3]*Mathematical Sciences Institute, Australian National University, Canberra ACT 0200, Australia*

A significant feature of the one-dimensional super Tonks-Girardeau gas is its metastable gas-like state with a stronger Fermi-like pressure than for free fermions which prevents a collapse of atoms. This naturally suggests a way to search for such strongly correlated behavior in systems of interacting fermions in one dimension. We thus show that the strongly attractive Fermi gas without polarization can be effectively described by a super Tonks-Girardeau gas composed of bosonic Fermi pairs with attractive pair-pair interaction. A natural description of such super Tonks-Girardeau gases is provided by Haldane generalized exclusion statistics. In particular, they are equivalent to ideal particles obeying more exclusive statistics than Fermi-Dirac statistics.

PACS number(s): 03.75.Ss, 05.30.Fk

Introduction. Recent experimental progress in manipulating cold atoms in reduced one-dimensional (1D) geometry [1–4] has stimulated intensive study of the physical properties of quantum gases, among which an important benchmark is the experimental realization of Tonks-Girardeau (TG) gases [3,4]. For the effective 1D systems, the effective 1D interactions can be tuned to reach the strongly interacting regime via Feshbach resonance or confinement-induced resonance [5]. The most recent experimental breakthroughs are the realization of a 1D super TG (sTG) gas of bosonic Cesium atoms [6] and a 1D spin-imbalanced Fermi gas of ^{6}Li atoms [7].

Whereas the TG gas describes the strongly repulsive Bose gas [8,9], the sTG gas describes a gas-like phase of the attractive Bose gas which can be described by a system of attractive hard rods [10,11]. The sTG gas state corresponds to a highly excited state in the integrable interacting Bose gas with attractive interaction [12]. Although the sTG state is a highly excited state which in principle should decay into the cluster ground state [13,14] of the attractive Bose gas, such a state is found to be realized and stabilized by switching the interactions between bosons from strongly repulsive to strongly attractive [6]. Due to the large kinetic energy inherited from the TG phase, the hard core behavior of the particles with Fermi-like pressure prevents the collapse of the sTG phase after the switch of interactions from repulsive to attractive [10,12].

In this Rapid Communication, we propose a scheme to realize the sTG gas in a Fermi system with attractive interactions. In contrast to the realization in the attractive interaction regime of the Bose gas [6], the sTG gas is composed of composite bosons which are bound pairs of fermions with opposite spins and thus is a true ground state (GS). We further demonstrate that such a sTG gas is identical to a system of ideal particles obeying Haldane generalized exclusion statistics (GES) [15] where the particles and holes are not equally weighted. In this sense, sTG and Fermi gases may also provide insight into the conceptual understanding of Haldane GES, which may possibly be counted by manipulating ultracold atoms.

Attractive fermion model. We consider a system composed of two hyperfine components with identical particle numbers $N_\uparrow = N_\downarrow = N/2$ in an elongated potential trap with $\omega_\perp \gg \omega_x$, where ω_x and $\omega_\perp \equiv \omega_y = \omega_z$ are angular frequencies in the axial and radial directions respectively. N is the total number of fermions. Under the condition $\omega_\perp/\omega_x \gg N$, such Fermi systems are dynamically described by an effective 1D Hamiltonian

$$H = \sum_{i=1}^{N} -\frac{\hbar^2}{2m_F} \frac{\partial^2}{\partial x_i^2} + g_F \sum_{i<j} \delta(x_i - x_j), \qquad (1)$$

where $g_F = -2\hbar^2/(m_F a_{\rm 1D}^F)$ is the effective 1D interaction strength related to the three-dimensional s-wave scattering length a_s^F by [5] $a_{\rm 1D}^F = -l_\perp(\frac{l_\perp}{a_s^F} - \frac{|\zeta(1/2)|}{\sqrt{2}})$ with $l_\perp = \sqrt{\hbar/m\omega_\perp}$ the characteristic oscillator length in the radial direction.

The eigenvalues of Hamiltonian (1) are given by $E = \frac{\hbar^2}{2m_F} \sum_{j=1}^{N} k_j^2$ with k_j determined by the Bethe ansatz equations (BAE) [16,17]

$$\exp(ik_j L) = \prod_{\alpha=1}^{M} \frac{k_j - \Lambda_\alpha + ic_F/2}{k_j - \Lambda_\alpha - ic_F/2},$$

$$\prod_{j=1}^{N} \frac{\Lambda_\alpha - k_j + ic_F/2}{\Lambda_\alpha - k_j - ic_F/2} = -\prod_{\beta=1}^{M} \frac{\Lambda_\alpha - \Lambda_\beta + ic_F}{\Lambda_\alpha - \Lambda_\beta - ic_F},$$

where $c_F = m_F g_F/\hbar^2 = -2/a_{\rm 1D}^F$, $j = 1, \ldots, N$, $\alpha = 1, \ldots, M$ and $M = N/2$ is the number of fermions with spin down.

This interacting fermion model has been widely studied (see, e.g., Refs. [18–25] and references therein). For strongly attractive interaction, i.e., $L|c_F| \gg 1$, the GS solutions of the BAE correspond to $M = N/2$ pairs of neutral charge bound states with $k_\alpha = \Lambda_\alpha \pm ic_F/2 + O(\delta)$ for $\alpha = 1, \ldots, M$. Here all Λ's are real and δ is a very small number of order $\exp(-L|c_F|)$ [26]. The BAE thus reduce to

$$\exp(2i\Lambda_\alpha L) = -\prod_{\beta=1}^{M} \frac{\Lambda_\alpha - \Lambda_\beta + ic_F}{\Lambda_\alpha - \Lambda_\beta - ic_F}. \qquad (2)$$

The eigenvalues of Hamiltonian (1) are given by $E = -M\epsilon_b + \frac{\hbar^2}{2m_F} \sum_{\alpha=1}^{M} 2\Lambda_\alpha^2$ where the binding energy $\epsilon_b = (\hbar^2/2m_F)c_F^2/2$, which characterizes internal energy and the other energy terms include the kinetic energy of the bound pairs and marginally interacting energy produced from pair-pair scattering in the strongly attractive interaction limit. In this limit and in the absence of an external field, we may subtract the binding energy from the energy, i.e.,

$$E_0^F = E + M\epsilon_b = \frac{\hbar^2}{2m_F} \sum_{\alpha=1}^{M} 2\Lambda_\alpha^2. \quad (3)$$

For strong coupling, the explicit Λ's follow from the BAE (2), i.e., $\Lambda_m \approx \frac{(2m+1)\pi}{2L}(1 - \frac{M}{L|c_F|})^{-1}$ (up to order $1/c_F^2$), with $m = -M/2, -M/2 + 1, \ldots, M/2 - 1$. Here we assume M is even. The GS energy follows as [21,22]

$$E_0^F \approx \frac{\hbar^2}{2m_F} \frac{1}{6} M(M^2 - 1) \frac{\pi^2}{L^2} \left(1 - \frac{M}{L|c_F|}\right)^{-2}. \quad (4)$$

Equivalence to a sTG gas. On the other hand, the 1D interacting Bose gas composed of N_B bosons is described by the Hamiltonian

$$H = \sum_{i=1}^{N_B} -\frac{\hbar^2}{2m_B} \frac{\partial^2}{\partial x_i^2} + g_B \sum_{i<j} \delta(x_i - x_j), \quad (5)$$

with $g_B = -2\hbar^2/(m_B a_{1d}^B)$. The energy eigenvalues are given in terms of the quasimomenta k_j by

$$E = \frac{\hbar^2}{2m_B} \sum_{j=1}^{N_B} k_j^2, \quad (6)$$

which satisfy the BAE [9]

$$\exp(ik_j L) = -\prod_{l=1}^{N_B} \frac{k_j - k_l + ic_B}{k_j - k_l - ic_B}, \quad (7)$$

with $c_B = m_B g_B/\hbar^2 = -2/a_{1D}^B$.

In the TG regime $(c_B \to \infty)$ the quasimomenta $k_m \approx \frac{(2m+1)\pi}{L}(1 + \frac{2N_B}{L|c_B|})^{-1}$, with $m = -N_B/2, -N_B/2 + 1, \ldots, N_B/2 - 1$. Here N_B is even. The GS energy of the strongly repulsive Bose gas in the TG regime (up to order $1/c_B^2$) is given by

$$E_{TG} \approx \frac{\hbar^2}{2m_B} \frac{1}{3} N_B (N_B^2 - 1) \frac{\pi^2}{L^2} \left(1 + \frac{2N_B}{L|c_B|}\right)^{-2}. \quad (8)$$

For attractive interaction $c_B < 0$, the GS solution for the BAE (7) is an N-string solution and the GS is described by a cluster state [13,14] with energy $E_0 = -\frac{1}{12}c_B^2 N_B(N_B^2 - 1)$. We note that the BAE (7) still have real solutions for $c_B < 0$, which obviously correspond to highly excited states. Solving the BAE (7) gives an explicit form for a gas-like highly excited state with $k_m \approx \frac{(2m+1)\pi}{L}(1 - \frac{2N_B}{L|c_B|})^{-1}$, where $m = -N_B/2, -N_B/2 + 1, \ldots, N_B/2 - 1$. Here N_B is even. In the strongly attractive region $(c_B \to -\infty)$, the energy of the sTG gas state follows as [12]

$$E_{STG} \approx \frac{\hbar^2}{2m_B} \frac{1}{3} N_B (N_B^2 - 1) \frac{\pi^2}{L^2} \left(1 - \frac{2N_B}{L|c_B|}\right)^{-2}. \quad (9)$$
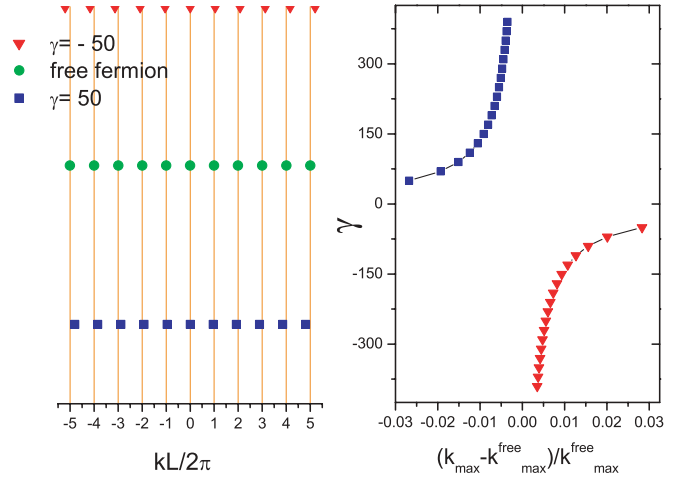

FIG. 1. (Color online) Quasimomentum distribution for the GS of the repulsive Bose gas with $\gamma = 50$ and the sTG gas phase of the attractive Bose gas with $\gamma = -50$ for $N_B = 11$ (left). The deviation from the free fermion distribution versus γ (right). Here $\gamma = c_B/n$.

Comparing Eqs. (4) and (9), it is clear they are identical if $c_B = 2c_F$, $N_B = M = N/2$ and $m_B = 2m_F$ (see also Ref. [21]). Since the bound pair formed by two fermions with opposite spin has a mass $m_B = 2m_F$, we can conclude that the M bound pairs are equivalently described by the sTG phase of the interacting Bose gas with the effective 1D scattering length

$$a_{1D}^B = \frac{1}{2} a_{1D}^F. \quad (10)$$

We note that relation (10), obtained by an exact mapping based on the exact many-body solutions, is consistent with that obtained by solving the four-body problem [20].

The mapping between the GS of the attractive Fermi gas and the sTG phase of the attractive Bose gas is exact and does not rely on the strong interaction expansion. In fact, substituting $c_F = c_B/2$ into BAE (2) and making a replacement $2\Lambda_\alpha = k_\alpha$, one finds that BAE (2) are identical to BAE (7) and the energy (3) is identical to the energy (6). To give a concrete example, we show the solutions of the BAE (2) and (7) in Fig. 1. For $|\gamma| = 50$, the roots of the sTG gas and strongly repulsive Bose gas are very close to the momentum distributions of free fermions, but on opposite sides of the free fermion distribution. With $|\gamma| \to \infty$, they approach the free fermion distribution. In Fig. 2, we show the GS energy for the repulsive Bose gas, the eigenenergy for the sTG gas phase of the attractive Bose gas, and the GS energy E_0^F for the attractive Fermi gas for different values of γ. It is clear that the subtracted GS energy E_0^F for the attractive Fermi gas is identical to the eigenenergy for the corresponding sTG gas phase with $m_B = 2m_F$, instead of the mass for the Bose gas.

The above conclusion also holds true in the thermodynamic limit $M, L \to \infty$ with $n = M/L$ finite, in which the GS energy (3) can be expressed in the form of the Gaudin integral equations [17]. The Gaudin equations for attractive fermions coincide exactly with the integral equation form of the sTG phase—they do not match the Lieb-Liniger equations for the Bose gas. We note that this mismatch in the sign of the integral equations for attractive fermions and repulsive bosons was already noted [19]. However, the "wrong" sign was argued to be irrelevant in the strong coupling limit. Here we recognize
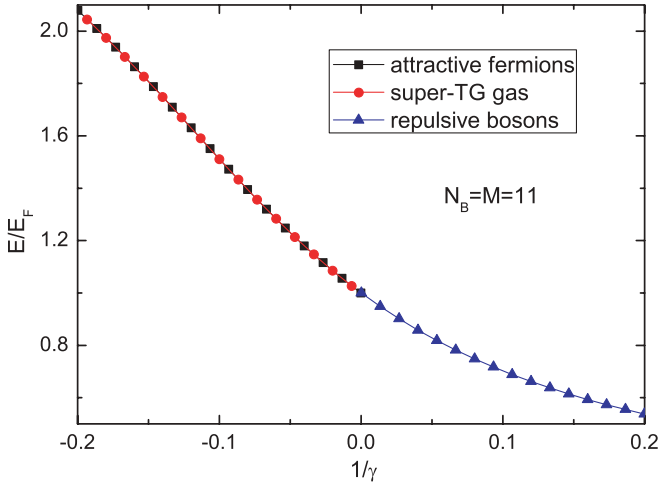
FIG. 2. (Color online) GS energy for the repulsive Bose gas (triangles), the eigenenergy for the sTG gas phase of the attractive Bose gas (dots), and the GS energy E_0^F for the attractive Fermi gas (squares) versus $1/\gamma$. E_F represents the GS energy of the TG gas with $\gamma \to \infty$.

that the GS of strongly attractive fermions shares the same signature as the sTG phase of the attractive Bose gas.

The low energy physics of 1D interacting bosons can be described by Tomonaga-Luttinger liquid (TLL) theory (see [27] for a review). The TG gas, which describes the strongly repulsive phase, corresponds to a TLL with $K > 1$ ($K \approx (1 + 2/|\gamma|)^2$) which characterizes the correlation length, e.g., the one-body correlation function $g^{(1)} = \langle \Psi^\dagger(x)\Psi(0)\rangle \propto 1/x^{\frac{1}{2K}}$. The sTG phase corresponds to a highly excited gas-like state where the particles are strongly correlated. This strongly collective behavior may be phenomenologically described by the TLL parameter $K \approx (1 - 2/|\gamma|)^2$ in the strongly interacting limit [12], which is smaller than 1. Consequently, the paired state of the Fermi gas is also described by a TLL with $K < 1$. In general, a system with $K < 1$ sometimes shows CDW quasi-order, making the system a quasi-supersolid [28]. However, we notice that the quasi-supersolid phase generally appears in lattice systems [28,29] with long range interactions. For a continuum system with only short range interactions, the quasi-supersolid phase or CDW order may be hard to realize in general, in contrast to other ultra-cold atomic systems in optical lattices [28,29].

Haldane exclusion statistics. Cooperative and collective behavior are significant features of many-body physics. In 1D pairwise dynamical interaction between identical particles is inextricably related to their statistical interaction. In particular, coherence between dynamical interaction and statistical interaction results in transmutation between these two types of interactions [30]. This can be seen from the equivalence between the 1D Bose gas and Haldane GES [15]. This equivalence was set up via an exact mapping

$$\alpha_{ij} := \alpha(k, k') = \delta(k, k') - \frac{1}{2\pi}\theta(k - k'), \qquad (11)$$

between the Bethe ansatz function $\theta(k) = 2c/(c^2 + k^2)$ and the GES parameter α [31]. In general, GES (11) for the 1D Bose gas is mutual statistics, i.e., $\alpha(k, k')$ depends on all of the other

quasimomenta when moving one particle away from the GS. Importantly, for the special case of strongly interacting bosons in 1D, Haldane GES [15] gives a quantitative description of the fermionization process where the parameter $\alpha_{TG} = (1 + \frac{2N_B}{|c_B|})^{-1} < 1$ is nonmutual [30]. In this case, the bosons are strongly correlated and behave like identical particles with GES α_{TG}. Here we further remark that for attractive bosons the GES description is not valid due to the existence of string solutions to the BAE. However, we may view all real Bethe ansatz roots as a GES distribution. In particular, from the set of quasimomenta $\{k_m \approx \frac{(2m+1)\pi}{L}(1 - \frac{2N_B}{L|c_B|})^{-1}\}$ of the sTG state we conceive that the minimum of separation in momentum space is larger than that of free fermions. In general, the momentum separation for identical particles with GES is given by $\Delta k_j \equiv 2\pi(\alpha + \ell)$ [32], where ℓ can be an arbitrary integer. For free fermions the minimum separation of the momentum is $2\pi/L$ with $\alpha = 1$. This minimum α naturally results in unequal weights for particle density $\rho(k)$ and hole density $\rho_h(k)$ distributions. We understand that for the sTG gas and the TG gas α number of bosons removed from the GS creates one hole, i.e.,

$$2\pi[\alpha\rho(k) + \rho_h(k)] \approx 1, \qquad (12)$$

with $\alpha = \alpha_{TG}$ or α_{sTG}. This gives the Haldane GES description with nonmutual GES. In this sense the recent experimental measurements in a 1D sTG gas of Cesium atoms [6] may also provide a measure of Fermi-like pressure induced from the GES parameter $\alpha_{STG} = (1 - \frac{2N_B}{|c_B|})^{-1}$ which is greater than the pure Fermi statistics value $\alpha = 1$.

For strongly attractive fermions in the absence of an external field, the neutral charge bound pairs become bosonic hard-core bosons with nonmutual GES statistics $\alpha_F = (1 - \frac{M}{|c_F|})^{-1}$ [25]. It is clearly seen that the GES parameters α_{STG} for the sTG gas and α_F for bound pairs of fermions are equivalent under the mapping $c_B = 2c_F$, $N_B = M = N/2$. The nonmutual GES for the TG gas, sTG gas and strongly attractive fermions can be unified by the most probable distribution $n(\epsilon)$

$$n(\epsilon) = \frac{1}{\alpha + w(\epsilon)}, \qquad (13)$$

where the function $w(\epsilon)$ satisfies the equation

$$w^\alpha(\epsilon)[1 + w(\epsilon)]^{1-\alpha} = e^{\epsilon - \mu/K_B T}, \qquad (14)$$

with μ the Fermi-like cut-off energy. Here we can easily see that for $\alpha = 0$ and $\alpha = 1$ the most probable distribution $n(\epsilon)$ (13) reduces to Bose-Einstein statistics and Fermi-Dirac statistics, respectively.

Now for TG and sTG bosons we have $N_B = \int_0^\infty d\epsilon G_B(\epsilon)n(\epsilon)$ and $E_B = \int_0^\infty d\epsilon G_B(\epsilon)n(\epsilon)\epsilon$ with density of states $G_B(\epsilon) = L/\sqrt{2\pi^2\hbar^2\epsilon/m_B}$. On the other hand, for attractive fermions $N_F = 2\int_0^\infty d\epsilon G_F(\epsilon)n(\epsilon)$ and $E_0^F = 2\int_0^\infty d\epsilon G_F(\epsilon)n(\epsilon)\epsilon$ with pair state density $G_F(\epsilon) = L/\sqrt{\pi^2\hbar^2\epsilon/m_F}$. For zero temperature, the GS energies of the TG gas and strongly attractive fermions are easily obtained through their nonmutual GES (13), along with the excited state energy for the sTG gas. The sTG gas result (9) can also be obtained from the minima of separation in quasimomentum space derived from GES. The GES approach provides an

alternative way to describe the thermodynamics of these models.

In summary, we have studied the equivalence between the GS of the strongly attractive Fermi gas and the sTG gas. We have shown that Haldane GES provides a natural description of these strongly correlated states. By comparing strongly attractive fermions with the Bose gas, we find that the bound Fermi pairs formed in the strongly attractive regime should be described by the sTG phase of the LL model of attractive bosons, rather than the LL model of repulsive bosons. This finding suggests that we can realize the sTG gas by preparing a 1D Fermi gas in the strongly attractive regime. Since the Fermi pairs are unbreakable in the strongly attractive limit, such a state is expected to be very stable. Moreover, our results suggest that experimental observation of Haldane statistics can be done by detecting the breathing mode of the attractive Fermi gas without polarization and comparing with the result obtained from the integrable anyon model with GES parameter α as fitting parameter [30].

[1] A. Görlitz *et al.*, Phys. Rev. Lett. **87**, 130402 (2001).

[2] H. Moritz, T. Stoferle, M. Kohl, and T. Esslinger, Phys. Rev. Lett. **91**, 250402 (2003); T. Stöferle, H. Moritz, C. Schori, M. Kohl, and T. Esslinger, *ibid.* **92**, 130403 (2004); H. Moritz, T. Stoferle, K. Guenter, M. Kohl, and T. Esslinger, *ibid.* **94**, 210401 (2005).

[3] B. Paredes *et al.*, Nature (London) **429**, 277 (2004).

[4] T. Kinoshita *et al.*, Science **305**, 1125 (2004).

[5] M. Olshanii, Phys. Rev. Lett. **81**, 938 (1998).

[6] E. Haller *et al.*, Science **325**, 1224 (2009).

[7] Y. Liao *et al.*, e-print arXiv:0912.0092.

[8] M. D. Girardeau, J. Math. Phys. **1**, 516 (1960).

[9] E. H. Lieb and W. Liniger, Phys. Rev. **130**, 1605 (1963).

[10] G. E. Astrakharchik, J. Boronat, J. Casulleras, and S. Giorgini, Phys. Rev. Lett. **95**, 190407 (2005).

[11] G. E. Astrakharchik, D. Blume, S. Giorgini, and B. E. Granger, Phys. Rev. Lett. **92**, 030402 (2004).

[12] M. T. Batchelor *et al.*, J. Stat. Mech. (**2005**) L10001.

[13] J. B. McGuire, J. Math. Phys. **6**, 432 (1965).

[14] E. Tempfli, S. Zöllner, and P. Schmelcher, New J. Phys. **10**, 103021 (2008).

[15] F. D. M. Haldane, Phys. Rev. Lett. **67**, 937 (1991).

[16] C. N. Yang, Phys. Rev. Lett. **19**, 1312 (1967).

[17] M. Gaudin, Phys. Lett. A **24**, 55 (1967).

[18] J. N. Fuchs, A. Recati, and W. Zwerger, Phys. Rev. Lett. **93**, 090408 (2004).

[19] I. V. Tokatly, Phys. Rev. Lett. **93**, 090405 (2004).

[20] C. Mora, A. Komnik, R. Egger, and A. O. Gogolin, Phys. Rev. Lett. **95**, 080403 (2005).

[21] T. Iida and M. Wadati, J. Phys. Soc. Jpn. **74**, 1724 (2005).

[22] M. T. Batchelor *et al.*, J. Phys. Conf. Ser. **42**, 5 (2006).

[23] G. Orso, Phys. Rev. Lett. **98**, 070402 (2007).

[24] H. Hu, X.-J. Liu, and P. D. Drummond, Phys. Rev. Lett. **98**, 070403 (2007).

[25] X.-W. Guan, M.T. Batchelor, C. Lee, and M. Bortz, Phys. Rev. B **76**, 085120 (2007).

[26] M. Takahashi, *Thermodynamic of One-Dimensional Solvable Models* (Cambridge University Press, Cambridge, 1999).

[27] M. A. Cazalilla, J. Phys. B **37**, S1 (2004).

[28] G. G. Batrouni, F. Hebert, and R. T. Scalettar, Phys. Rev. Lett. **97**, 087209 (2006).

[29] D. Heidarian and K. Damle, Phys. Rev. Lett. **95**, 127206 (2005); R. G. Melko, A. Paramekanti, A. A. Burkov, A. Vishwanath, D. N. Sheng, and L. Balents, *ibid.* **95**, 127207 (2005); P. Sengupta, L. P. Pryadko, F. Alet, M. Troyer, and G. Schmid, *ibid.* **94**, 207202 (2005); S. Wessel and M. Troyer, *ibid.* **95**, 127205 (2005); V. W. Scarola, E. Demler, and S. Das Sarma, Phys. Rev. A **73**, 051601(R) (2006).

[30] M. T. Batchelor and X.-W. Guan, Phys. Rev. B **74**, 195121 (2006); Laser Physics Letters **4**, 77 (2006).

[31] Y.-S. Wu, Phys. Rev. Lett. **73**, 922 (1994); D. Bernard and Y.-S. Wu, e-print arXiv:cond-mat/9404025.

[32] Z. N. C. Ha, Phys. Rev. Lett. **73**, 1574 (1994).