

Super-Tonks-Girardeau Gas of Spin-1/2 Interacting Fermions

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(Received 6 May 2010; revised manuscript received 30 August 2010; published 19 October 2010)

Fermi gases confined in tight one-dimensional waveguides form two-particle bound states of atoms in the presence of a strongly attractive interaction. Based on the exact solution of the one-dimensional spin-1/2 interacting Fermi gas, we demonstrate that a stable excited state with no pairing between attractive fermionic atoms can be realized by a sudden switch of interaction from the strongly repulsive regime to strongly attractive regime. Such a state is an exact fermionic analog of the experimentally observed super-Tonks-Girardeau state of bosonic Cesium atoms [*Science* **325**, 1224 (2009)] and should be possible to be observed by the experiment. The frequency of the lowest breathing mode of the fermionic super-Tonks-Girardeau gas is calculated as a function of the interaction strength, which could be used as a detectable signature for the experimental observation.

DOI: 10.1103/PhysRevLett.105.175301

PACS numbers: 67.85.-d, 03.75.Ss

Introduction.—Exploring new quantum phases and understanding the striking consequences of correlation in strongly interacting atomic gases are at the frontier of current research in condensed matter physics and cold atom physics [1,2]. In recent years, remarkable progress has been made in the experimental realization of fundamental many-body model Hamiltonians, such as the Hubbard model [3] and the Tonks-Girardeau (TG) gas [4,5], with unprecedented tunability. In general, attentions are devoted to the exotic properties of ground states and low-excited states. A recent experimental breakthrough is the realization of the super Tonks-Girardeau (STG) gas of bosonic cesium atoms [6], which is a stable highly excited state of interacting Bose gas [7–9]. In the experiment [6], a one-dimensional (1D) Bose gas was initially prepared in the strongly repulsive TG regime, and then the STG gas was obtained by suddenly switching the interaction from strongly repulsive to attractive regime. A striking feature of the STG gas is its counterintuitive stability against collapsing to its cluster ground state even in the presence of strongly attractive interactions.

So far, the experimental study [6] and most of the theoretical works [7–12] on the STG gas have focused on the bosonic system. In this work, we study the possible realization of the Fermi super Tonks-Girardeau (FSTG) state in a 1D Fermi gas. As the 1D Fermi gas with tunable interaction strengths has already been experimentally realized [13], it is promising to directly observe the STG state in 1D Fermi gases. Stimulated by the experiment of the Bose STG gas, we first prepare a strongly repulsive spin-1/2 Fermi gas, and then suddenly switch the interaction to the strongly attractive regime. By this way, we can access a stable highly excited state of the attractive Fermi gas which does not fall into its attractive ground state. In the strongly attractive limit, atoms with different spins form tightly bound fermion pairs [14–16]. It has been shown that the ground state of a 1D attractive spin-balanced Fermi gas is

effectively described by the STG state of bosonic pairs of fermions with attractive pair-pair interaction [16]. The FSTG state being studied in the present work is essentially the lowest gaslike excited state composed of unpaired fermions which is totally different from the bosonic STG state composed of tightly bound fermion pair proposed in Ref. [16].

Interacting Fermi model.—We consider a system of $N = N_{\uparrow} + N_{\downarrow}$ spin-1/2 fermions in a tightly confined waveguide described by the effective 1D Hamiltonian

$$H = -\frac{\hbar^2}{2m} \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + g_{1d} \sum_{i<j} \delta(x_i - x_j), \quad (1)$$

where $g_{1d} = -2\hbar^2/(ma_{1d}) = \hbar^2 c/m$ is the effective 1D interaction strength and a_{1D} the effective 1D scattering length [17]. Without loss of generality, we assume that $N_{\downarrow} \leq N_{\uparrow}$. The 1D interacting spin-1/2 Fermi gas is only solvable for the homogenous case [14]. However, in the infinitely repulsive limit, a generalization of Bose-Fermi mapping [18] to the spin-1/2 Fermi system [19,20] allows us to construct analytically exact solution of 1D Fermi gases even in trap potentials. In this work, we focus on the homogenous system which can be exactly solved by the Bethe-ansatz (BA) method. The trapped system can be studied in the scheme of the local density approximation (LDA).

The model (1) is exactly solved by the BA method [14] with the BA wave function

$$\begin{aligned} \varphi(x_1, \dots, x_N) = & \sum_Q \sum_P \theta(x_{Q1} \leq \dots \leq x_{QN}) \\ & \times [Q, P] \exp \left[i \sum_{j=1}^N k_{Pj} x_{Qj} \right], \quad (2) \end{aligned}$$

where k_i represent quasimomenta, P and Q represent permutations of k_i and x_i , respectively. For the

eigenstate with the total spin $S = N/2 - M$ ($M = N_{\downarrow}$), the coefficient $[Q, P]$ can be explicitly expressed as $[Q, P] = \sum_{T=1}^{C_M^N} \Phi(y_{T_1}, y_{T_2}, \dots, y_{T_M}; P) \prod_{j=1}^M \chi_{y_{T_j}, \downarrow} \prod_{x_i \neq y_{T_j}} \chi_{x_i, \uparrow}$, where $\chi_{x_i, \uparrow}$ ($\chi_{y_j, \downarrow}$) denotes the up (down)-spin, T is a combination of M down-spins in N particles, $\{y_{T_j}\}$ are M elements of $T\{x_i\}$, and $\Phi(y_{T_1}, y_{T_2}, \dots, y_{T_M}; P) = \sum_R A(R) \times \prod_{j=1}^M F_P(\Lambda_{R_j}, y_{T_j})$ with R being the permutations of Λ_s , $A(R) = \epsilon(R) \prod_{j < l} (\Lambda_{R_j} - \Lambda_{R_l} - ic)$, and $F_P(\Lambda_{R_j}, y_{T_j}) = \prod_{i=y_{T_j}-1}^{y_{T_j}} (k_{P_i} - \Lambda_{R_j} + ic/2) \prod_{i=y_{T_j}+1}^N (k_{P_i} - \Lambda_{R_j} - ic/2)$. The parameters k_j and Λ_α are determined by the Bethe-ansatz equations (BAEs) [14]:

$$k_j L = 2\pi I_j - 2 \sum_{\alpha=1}^M \tan^{-1} \left(\frac{k_j - \Lambda_\alpha}{c/2} \right), \quad (3)$$

$$\sum_{j=1}^N 2 \tan^{-1} \left(\frac{\Lambda_\alpha - k_j}{c/2} \right) = 2\pi J_\alpha + 2 \sum_{\beta=1}^M \tan^{-1} \left(\frac{\Lambda_\alpha - \Lambda_\beta}{c} \right). \quad (4)$$

The eigenenergies are given by $E = \frac{\hbar^2}{2m} \sum_j k_j^2$. Here both k_j and Λ_α are real numbers if $c > 0$. The ground state solution corresponds to $I_j = (N+1)/2 - j$ and $J_\alpha = (M+1)/2 - \alpha$. In the limit of $cL \gg 1$, Λ_α are proportional to c , but k_j remain finite, therefore the quasimomenta can be given approximately

$$k_j L = 2\pi I_j - \zeta \frac{k_j}{|c|} + O(|c|^{-3}) \quad (5)$$

with $\zeta = \sum_{\alpha=1}^M \frac{1}{(\Lambda_\alpha/c)^2 + 1/4}$. It follows that the ground energy in the strongly repulsive limit reads

$$E_{\text{FTG}} = \frac{\hbar^2}{2m} \frac{\pi^2}{3L^2} N(N^2 - 1) \left(1 + \frac{\zeta}{L|c|} \right)^{-2} + O(|c|^{-3}), \quad (6)$$

which is consistent with the result in Ref. [21] up to order of c^{-1} . In the limit of $c \rightarrow \infty$, the ground energy is identical to that of a polarized N -fermion system.

FSTG state.—If the interaction is attractive, the ground state is composed of $N - 2M$ real k_i and $2M$ complex ones. In the limit $-cL \gg 1$, the complex solutions take the 2-string form: $k_\alpha \approx \Lambda_\alpha + \frac{\zeta}{2}i$, and $k_{M+\alpha} \approx \Lambda_\alpha - \frac{\zeta}{2}i$. Except the complex solutions, the BAEs also have real solutions for $c < 0$, which, however, correspond to some highly excited states of attractive Fermi systems. The FSTG state corresponds to the lowest real solutions of BAEs (3) and (4) with $c < 0$. In this case, Λ_α go infinite and k_j remain finite with $|c|L \rightarrow \infty$, thus the momenta are given by

$$k_j L = 2\pi I_j + \zeta \frac{k_j}{|c|} + O(|c|^{-3}). \quad (7)$$

Despite the ζ in Eq. (7) having the same form as in Eq. (5), generally $\zeta(c) \neq \zeta(-c)$ since the solutions Λ_α of Eq. (4) are not symmetric for c and $-c$. However, in the strong coupling limit, up to order of c^{-1} Eq. (4) becomes $2N \tan^{-1}(\frac{\Lambda_\alpha}{c/2}) = 2\pi J_\alpha + 2 \sum_{\beta=1}^M \tan^{-1}(\frac{\Lambda_\alpha - \Lambda_\beta}{c}) + O(|c|^{-2})$, which is invariant under the operation $P: \{c \rightarrow -c, \Lambda_\alpha \rightarrow -\Lambda_\alpha\}$. Therefore, we have $\zeta(c) = \zeta(-c)$ up to the order of c^{-2} . The energy of the FSTG gas in the strongly attractive limit is thus given by

$$E_{\text{FSTG}} = \frac{\hbar^2}{2m} \frac{\pi^2}{3L^2} N(N^2 - 1) \left(1 - \frac{\zeta}{L|c|} \right)^{-2} + O(|c|^{-3}). \quad (8)$$

In the limit of $|c| \rightarrow \infty$, we have $E_{\text{FSTG}} = E_{\text{FTG}}$ and $k_j = I_j 2\pi/L$ for both the Fermi TG and the FSTG gas. In Fig. 1(a), for an example system with $N = 10$ and $M = 5$, we show the BAE solutions of k_j for the repulsive Fermi gas and the attractive FSTG gas with different values of $\gamma = c/\rho$, where $\rho = N/L$ is the particle density. The quasimomentum distributions for the repulsive Fermi gas and the STG gas approach the same limit from different sides when $|\gamma|$ goes infinite. Correspondingly, E_{FSTG} and E_{FTG} also approach the same limit $E_{\text{inf}} = \frac{\hbar^2}{2m} \frac{\pi^2}{3L^2} N(N^2 - 1)$ as shown in Fig. 1(b).

The FSTG state can be achieved through a similar sudden switch as in Ref. [6]. The system is first prepared at the ground state in the strongly repulsive regime, i.e., $|\Psi(t=0)\rangle = |\varphi_0(c)\rangle$. After a sudden switch into the strongly attractive regime with $c' < 0$, the wave function is given by $|\Psi(t)\rangle = e^{-iHt} |\varphi_0(c)\rangle = \sum_i e^{-iE_i t} \alpha_i |\varphi_i(c')\rangle$, where $\varphi_i(c')$ is the i th eigenstate of the Hamiltonian with parameter c' and $\alpha_i = \langle \varphi_i(c') | \varphi_0(c) \rangle$. The probability for the system staying in a state $|\varphi_i(c')\rangle$ is given by $|\alpha_i|^2 = |\langle \varphi_i(c') | \varphi_0(c) \rangle|^2$. We note that the wave-functions $\varphi_{\text{STG}}(c')$ and $\varphi_0(c)$ are identical when $c' = -\infty$ and $c = \infty$, and thus one can expect the probability of the system transforming from the Fermi TG gas to STG phase to be close to 1 for large $|c'|$ and $|c|$. In Fig. 2, we display transition

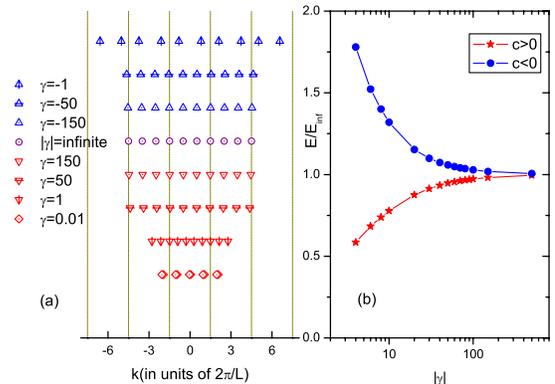


FIG. 1 (color online). (a) Quasimomentum distributions for the ground state of the repulsive Fermi gas and the FSTG state of the attractive Fermi gas with different values of γ . (b) The energies E_{FTG} (stars) and E_{FSTG} (dots) vs γ .

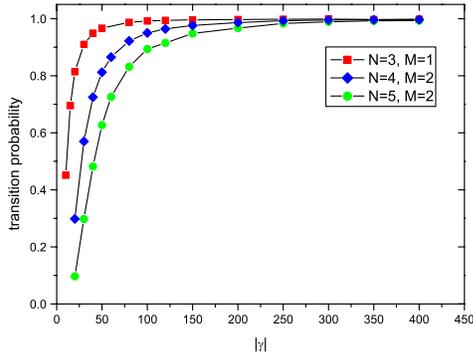


FIG. 2 (color online). Transition probabilities from the Fermi TG gas to FSTG phase for systems with $N = 3, 4, 5$.

probabilities from the initial ground state with $c > 0$ to the STG phase with $c' = -c$ for different-size systems. Our results show that the overlap of wave functions approaches the limit of 1 for large $|\gamma|$, although it decreases as the system size increases. In the strongly interacting limit, the transition probabilities can be approximately represented as $P(N, M, \gamma) = 1 - a(N, M)/\gamma^2$. The upper bound of the parameter $a(N, M)$ can be estimated as $a(N, M) \leq 64\pi^2 NM^2$ through the expansion of wave functions to order of $1/c$. The transition probability for a larger system is expected to approach 1 if $|\gamma| \gg MN^{1/2}$. However, the calculation for a large system becomes a very time-consuming task due to the calculation of multidimensional integrals.

In thermodynamic limit where N, M , and L go infinite but $\rho = N/L$ and $m = M/L$ remain finite, the BAEs can be expressed in the form of the coupled integral equations [14]. The energy per particle reads $\epsilon(\gamma) = \frac{\hbar^2}{2m} \rho^2 e(\gamma)$. For the case with $m = \rho/2 = \int_{-\infty}^{\infty} \sigma(\Lambda) d\Lambda$, ζ can be calculated via the integral $\zeta = \int_{-\infty}^{\infty} dx 4\sigma(x)/(x^2 + 1) = 2 \ln 2$, where $\sigma(x) = 1/[4 \cosh(\pi x/2)]$. Therefore, we can obtain the energy expansion $e_{\text{STG}}(\gamma) = \frac{\pi^2}{3} [1 + 4 \ln 2 |\gamma|^{-1} + 12 (\ln 2)^2 |\gamma|^{-2}] + O(\gamma^{-3})$ for large $|\gamma|$. The energy in the whole area can be calculated by numerical solving the coupled integral equations. Comparing the expansion result with the exact numerical result, we find that they

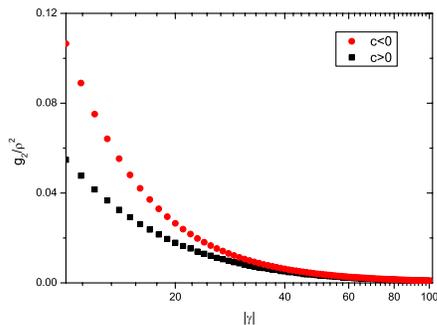


FIG. 3 (color online). Local correlation energy vs γ for the FSTG state and ground state of the repulsive Fermi gas.

agree very well in the regime of large γ , for example, $|e_{\text{expansion}} - e_{\text{num}}|/|e_{\text{num}}| < 10^{-3}$ for $\gamma = 25$. A characteristic of the STG gas is that it has stronger correlations than the TG gas. In the regime of $|\gamma| \gg 1$, the local two-particle correlation function $g_2(\gamma) = \rho^2 de(\gamma)/d\gamma$ can be directly derived from $e(\gamma)$, which gives $g_2(\gamma)_{\text{TG}}/\rho^2 \approx (4\pi^2/3)(\ln 2 |\gamma|^{-2} - 6(\ln 2)^2 |\gamma|^{-3})$, and $g_2(\gamma)_{\text{STG}}/\rho^2 \approx (4\pi^2/3)(\ln 2 |\gamma|^{-2} + 6(\ln 2)^2 |\gamma|^{-3})$. Thus, we have $g_2(-|\gamma|)_{\text{STG}} > g_2(|\gamma|)_{\text{TG}}$ as shown in Fig. 3.

In terms of terminologies of Tomonaga-Luttinger liquid (TLL) theory [22], the strongly repulsive phase of the spin-balanced Fermi gas corresponds to a TLL with the charge TLL parameter $K_c \approx (1 + 4 \ln 2/|\gamma|)/2 > 1/2$ [22]. The FSTG phase corresponds to a highly excited gaslike state where unpaired particles are strongly correlated. This strongly collective behavior may be phenomenologically described by $K_c \approx (1 - 4 \ln 2/|\gamma|)/2$ in the strongly interacting limit, which is smaller than $1/2$. For both cases, the spin TLL parameter $K_\sigma = 1$ due to spin-rotational invariance.

Degeneracy of the FSTG state.—In comparison to the Bose system, the ground state of a spin-1/2 system is highly degenerate in the TG limit due to the fact that states with different total spins have the same energy [20]. However, for a large but finite interaction strength the degeneracy is broken and the true ground state is the state with the lowest S . For the spin-1/2 system described by (1), one can understand this fact from the energy expression (6). The term ζ is M dependent ($M = N/2 - S$) and we have $\zeta(M_1) < \zeta(M_2)$ if $M_1 < M_2$, which leads to $E_{\text{FTG}}(S_2) < E_{\text{FTG}}(S_1)$ for $S_2 < S_1$. This is consistent with the Lieb-Mattis theorem [23]. The energy difference is proportional to $1/c$ and vanishes as $c \rightarrow \infty$. On the other hand, for the FSTG state we have $E_{\text{FSTG}}(S_2) > E_{\text{FSTG}}(S_1)$ for $S_2 < S_1$ according to Eq. (8), i.e., the FSTG state with the smaller S has higher energy. To give an example, we calculate the ground state energy and the energy of the FSTG state for a system with $N_\uparrow = N_\downarrow = 2$. As shown in Fig. 4, energies for states with different total spins approach the same limit of the polarized Fermi gas as $|c| \rightarrow \infty$.

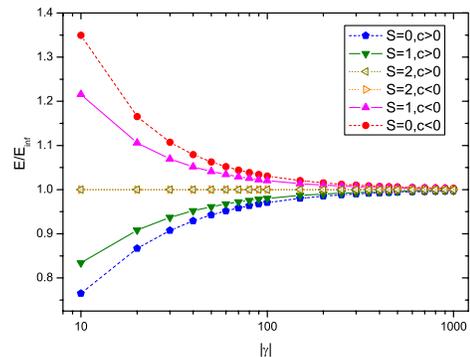


FIG. 4 (color online). Energy vs γ for states with different total spin S .

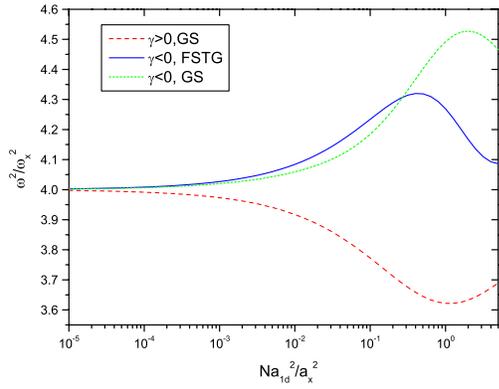


FIG. 5 (color online). Square of the lowest breathing mode frequency vs the interaction strength Na_{1d}^2/a_x^2 for the FSTG gas (solid line), the repulsive Fermi TG gas (dashed line) and the ground state of the attractive Fermi gas (dotted line).

Experimental detection.—To realize the FSTG gas, one can first tune the interaction of the 1D Fermi gas to the strongly repulsive regime by the Feshbach resonance [13], and then suddenly switch the interaction across the resonance point. Similar to the bosonic case, one can measure the frequency of the breath mode of the FSTG gas subjected to a weak harmonic confinement along the axial direction, which is sensitive to various regimes of interaction. For the Fermi gas in a harmonic trap with $V_{\text{ext}} = m\omega_x^2 x^2/2$, we can determine the density distribution of the STG gas within the LDA. According to the LDA, the system is in local equilibrium at each point x in the external trap. The density distribution of the FSTG gas is then obtained via the local equation of state $\mu_0 = \mu[\rho(x)] + V_{\text{ext}}(x)$ [24–26]. Here $\mu(\rho) = \partial_\rho[\rho\epsilon(\rho)]$ is the local chemical potential with $\epsilon(\rho)$ the energy density of the homogenous FSTG gas to be determined by numerically solving integral BA equations, and μ_0 is determined by the normalization condition $\int dx\rho(x) = N$. Following Refs. [25,26], we calculate the frequency of the lowest breathing mode from the mean square radius of the trapped FSTG gas via $\omega^2 = -2\langle x^2 \rangle / (d\langle x^2 \rangle / d\omega_x^2)$ with $\langle x^2 \rangle = \int \rho(x)x^2 dx / N$. The solid line in Fig. 5 shows the frequency of breathing mode of the attractive FSTG gas as a function of the interaction strength Na_{1d}^2/a_x^2 with the harmonic oscillator length $a_x = \sqrt{\hbar/m\omega_x}$. The frequency of the breath mode for the FSTG gas exhibits a peak with a maximum of ω^2/ω_x^2 about 4.3. We also give results of the repulsive Fermi TG gas (the dashed line) and the ground state of the attractive Fermi gas (the dotted line) for comparison. These results are essentially based on the sum-rule approach and provide generally an upper bound on the frequencies [25].

Summary.—In summary, we study the realization of the FSTG phase in the interacting spin-1/2 Fermi gas. Starting

from the ground state of a strongly repulsive Fermi gas, the FSTG state can be realized by a sudden switch of interaction to the strongly attractive regime. It is shown that the FSTG state is stable against forming pairing states even in the presence of the strongly attractive interaction between fermionic atoms with opposite spins. We also calculate the lowest breathing mode frequency of the FSTG gas which may be detected by the experiment.

We thank X.-W. Guan and X. Yin for helpful discussion. This work has been supported by the NSF of China under Grants No. 10974234 and No. 10821403, 973 grant and National Program for Basic Research of MOST.

While this work was being prepared for submission, a related manuscript appeared [27], in which the FSTG gas in a harmonic trap is studied.

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