Topological Superconductor to Anderson Localization Transition in One-Dimensional Incommensurate Lattices

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We study the competition of disorder and superconductivity for a one-dimensional p-wave superconductor in incommensurate potentials. With the increase in the strength of the incommensurate potential, the system undergoes a transition from a topological superconducting phase to a topologically trivial localized phase. The phase boundary is determined both numerically and analytically from various aspects and the topological superconducting phase is characterized by the presence of Majorana edge fermions in the system with open boundary conditions. We also calculate the topological Z_2 invariant of the bulk system and find it can be used to distinguish the different topological phases even for a disordered system.

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Introduction.—Topological superconductors (TSCs) have attracted intense recent studies, as they are promising candidates for the practical realization of Majorana fermions [1–7]. Among various proposals, the one-dimensional (1D) TSC in nanowires with strong spin-orbit interactions and proximity-induced superconductivity [6,7] provides experimental feasibility on the detection of Majorana fermions in hybrid superconductor-semiconductor wires [8–10], which has stimulated great enthusiasm in exploring the physical properties of topological superconductors. A key feature of a 1D TSC is the emergence of edge Majorana fermions (MFs) at ends of the superconducting (SC) wire as a result of bulk-boundary correspondence. A prototype model unveiling topological features of the 1D TSC is given by the effective spinless p-wave SC model studied originally by Kitaev [2].

As the TSC is protected by the particle-hole symmetry, the topological phase is expected to be immune to perturbations of weak disorder [11]. Nevertheless, a strong disorder may destroy the SC phase and induce a transition to the Anderson insulator. Localization in the 1D SC system in the presence of disorder has been an active research field in the past decades [12–15]. The theoretical studies have unveiled that the particle-hole symmetry in the SC system plays an important role in the problem of the Anderson localization [12]. Due to the existence of a finite SC gap, the interplay of disorder and superconductivity leads to a topological phase transition from the topological SC phase to a topologically trivial localized phase when the strength of disorder increases over a critical value.

So far, most theoretical work for the Anderson localization in 1D TSCs focuses on the random disorder [13–17]; disorder produced by incommensurate potentials has been of concern only very recently [18,19]. While Ref. [18] explores the TSC phase by tuning the chemical potential in a 1D quantum wire with spin-orbit interaction in proximity to a superconductor under incommensurate modulation, we focus our study on the transition from the TSC phase to Anderson localization purely induced by the incommensurate potential for a 1D p-wave superconductor system. In the absence of superconductivity, the localization transition driven by the incommensurate potential occurs at a finite disorder strength which can be exactly determined by a self-duality mapping [20], whereas an arbitrary weak random disorder induces the Anderson localization in one dimension. The incommensurate potential can now be engineered with ultracold atoms loaded in 1D bichromatic optical lattices [21], opening the experimental way to study the localization properties of quasiperiodic systems. In this Letter, we shall study the interplay of the incommensurate potential and topologically protected superconductivity in the 1D p-wave SC model and determine the phase boundary of the TSC to the localization transition exactly. The tunability of the incommensurate potential [21] provides a potential way to experimentally study the controllable disorder effect in TSCs realizable in cold atom systems [22].

Model of p-wave superconductor with incommensurate potential.—The 1D p-wave superconductor in the incommensurate lattices is described by the following Hamiltonian:

$$H = \sum_i [(-t\hat{c}_i^\dagger \hat{c}_{i+1} + \Delta \hat{c}_i \hat{c}_{i+1} + \text{H.c.}) + V_i \hat{n}_i], \quad (1)$$

where $\hat{n}_i = \hat{c}_i^\dagger \hat{c}_i$ is the particle number operator and $\hat{c}_i^\dagger$ ($\hat{c}_i$) is the creation (annihilation) operator of fermions. Here the nearest-neighbor hopping amplitude t and the p-wave pairing amplitude Δ are taken as real constants, whereas the incommensurate potential

$$V_i = V\cos(2\pi i\alpha) \tag{2}$$

varies at each lattice site with α being an irrational number and V the strength of the incommensurate potential. The model reduces to the Aubry-André model when $\Delta = 0$ [20], while the Hamiltonian describes the Kitaev p-wave SC model for $\alpha = 0$ [2]. For $\Delta = 0$, the system undergoes a delocalization to localization transition at $V = 2t$. On the other hand, the uniform p-wave SC system with $V_i = V$ undergoes a topological phase transition at $|V| = 2t$ with a topological nontrivial phase in the regime of $|V| < 2t$ characterized by the presence of edge MFs [2]. In this Letter, we shall study the interplay of the SC pairing Δ and the incommensurate potential and then determine the phase diagram of the system.

The Hamiltonian can be diagonalized by using the Bogoliubov–de Gennes (BdG) transformation [23,24]:

$$\eta_n^\dagger = \sum_{i=1}^{L}[u_{n,i}\hat{c}_i^\dagger + v_{n,i}\hat{c}_i], \tag{3}$$

where L is the number of lattice sites and $n = 1, \ldots, L$. Here $u_{n,i}$ and $v_{n,i}$ are chosen real. In terms of the operators η_n and $\eta_n^\dagger$, the diagonalized Hamiltonian is written as $H = \sum_{n=1}^{L}\Lambda_n(\eta_n^\dagger\eta_n - \frac{1}{2})$ with Λ_n being the spectrum of the single quasiparticles. The spectrum as well as $u_{n,i}$ and $v_{n,i}$ can be determined by solving the BdG equations,

$$\begin{pmatrix} \hat{h} & \hat{\Delta} \\ -\hat{\Delta} & -\hat{h} \end{pmatrix}\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \Lambda_n\begin{pmatrix} u_n \\ v_n \end{pmatrix}, \tag{4}$$

where $\hat{h}_{ij} = -t(\delta_{j,i+1} + \delta_{j,i-1}) + V_i\delta_{ji}$, $\hat{\Delta}_{ij} = -\Delta(\delta_{j,i+1} - \delta_{j,i-1})$, $u_n^T = (u_{n,1}, \ldots, u_{n,L})$, and $v_n^T = (v_{n,1}, \ldots, v_{n,L})$. The symmetry of the BdG equations implies $\eta_n(\Lambda_n) = \eta_n^\dagger(-\Lambda_n)$. The ground state of the system corresponds to the state with all negative quasiparticle energy levels filled. If the quasiparticle energies are arranged in ascending order, i.e., $\Lambda_i \leq \Lambda_{i+1}$, for $\Lambda_i > 0$, the gap of the system is just given by $\Delta_g = 2\Lambda_1$. In the following calculation, we shall set $t = 1$ as the energy unit.

Transition from SC phase to disorder phase.— Numerically solving Eqs. (4), we can obtain the whole spectrum of quasiparticles. In Fig. 1, we show the spectra for the case of $\alpha = (\sqrt{5} - 1)/2$ and $\Delta = 0.5$ under periodic boundary conditions (PBC). It is shown that there exists a regime with obvious nonzero gaps when V is smaller than a critical value V_c. When V exceeds the critical value, there is no obvious gap separating the negative and positive parts of the spectra. To see it more clearly, we show the variation of Δ_g versus V in the regime close to the transition point in Fig. 2(a). As shown in the figure, the gap vanishes at about $V_c = 3$ and the system opens a very narrow gap in the regime of $V > V_c$. We calculate the gap for systems with different Δ and find similar behavior: the gap reaches a minimum, which approaches zero in the limit

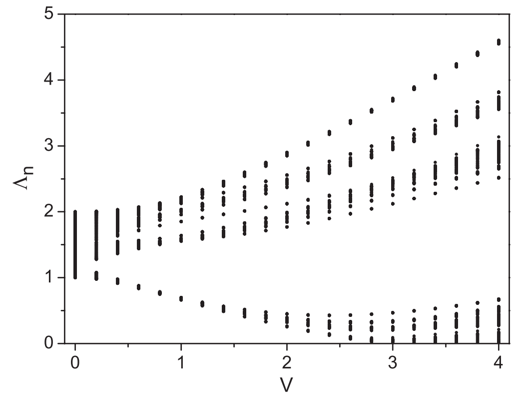

FIG. 1. Energy spectra of 1D p-wave superconductors with $\alpha = (\sqrt{5} - 1)/2$, $\Delta = 0.5$, and $L = 500$ under PBC.

of $L \to \infty$, at the transition point about $V_c = 2 + 2\Delta$ and there exits a very narrow gap when V exceeds the transition point. For cases with different irrational α, we find similar phenomena and the transition point does not depend on the specific choice of α (see the Supplemental Material [25]).

Observing that the p-wave fermion model corresponds to the transverse XY model with a randomly (irrationally) modulated transverse field [26,27], $\hat{H} = -\sum_i[J_x\sigma_i^x\sigma_{i+1}^x + J_y\sigma_i^y\sigma_{i+1}^y] + \sum_i h_i\sigma_i^z$, with the identification of $J_x = (t + \Delta)/2$, $J_y = (t - \Delta)/2$ and $h_i = -V_i/2$, we can identify the phase transition by calculating the correlation function $C_{ij} = \langle\sigma_i^x\sigma_j^x\rangle$. In the language of the quantum spin model, the ferromagnetic phase is characterized by the long-range order of the correlation function $\sigma_i^x\sigma_j^x\rangle_{|i-j|\to\infty} = A$ with A being a nonzero positive number. In the original fermion representation, $\sigma_i^x = (\hat{c}_i^\dagger + \hat{c}_i)\exp(-i\pi\sum_{j=1}^{i-1}\hat{c}_j^\dagger\hat{c}_j)$ takes a nonlocal form including a string product of fermion

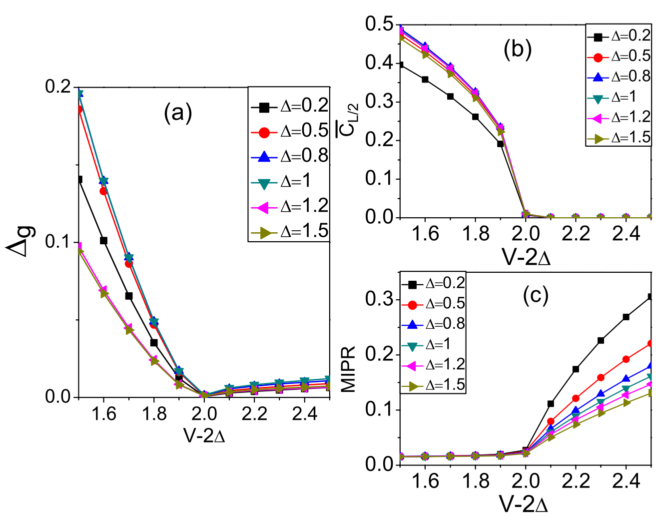

FIG. 2 (color online). (a) Energy gap Δ_g versus $V - 2\Delta$. (b) Average correlation function $\bar{C}_{L/2}$ versus $V - 2\Delta$. (c) MIPR versus $V - 2\Delta$ for the system with $\alpha = (\sqrt{5} - 1)/2$ and $L = 500$.

operators, and the correlation function $C_{ij} = \langle(\hat{c}_i^\dagger + \hat{c}_i) \times \exp(-i\pi \sum_{l=i}^{j} \hat{n}_l)(\hat{c}_j^\dagger + \hat{c}_j)\rangle$. In the presence of the disordered potential, the correlation function C_{ij} will oscillate and we define the average correlation function $\bar{C}_r = \sum_i C_{i,i+r}/L$. Then for a large system under PBC, the value of $\bar{C}_{L/2}$ can be used to distinguish the SC phase and the localized phase. The correlation function C_{ij} can be calculated by the exact numerical method described in Ref. [27]. In Fig. 2(b) we show the relation between $\bar{C}_{L/2}$ and V for systems with different Δ. Without the disordered potential, the correlation function $\bar{C}_{L/2}$ is a positive number and increases as Δ increases for $0 < \Delta < 1$, obtains its largest value $\bar{C}_{L/2} = 1$ at $\Delta = 1$, then decreases for $\Delta > 1$. As the strength of V increases, $\bar{C}_{L/2}$ decreases monotonically and approaches zero when $V - 2\Delta$ is about 2. When $V > 2 + 2\Delta$, the system loses the long-range order of the correlation function and the system is driven into the Anderson localized phase.

To characterize the localization transition, we define the quantity of the inverse participation ratio (IPR) as $P_n = \sum_{i=1}^{L}(u_{n,i}^4 + v_{n,i}^4)$, where $u_{n,j}$ and $v_{n,j}$ are the solution to the BdG equations and fulfil the normalization condition $\sum_i(u_{n,i}^2 + v_{n,i}^2) = 1$. The above definition can be viewed as an extension of the IPR for the case with $\Delta = 0$ [28,29]. For an extended state, $P_n \to 1/L$ and the IPR tends to zero for large L, whereas the IPR tends to a finite number for a localized state. Therefore, the IPR can be taken as a criterion to distinguish the extended states from the localized ones. Since the ground state is composed of states with all negative quasiparticle energy levels filled, we define the mean inverse participation ratio (MIPR) as $\mathrm{MIPR} = \sum_{n=1}^{L} P_n/L$ to characterize the localization of the ground state. As shown in Fig. 2(c), the MIPR increases monotonically with the increase of V. At $V = 2 + 2\Delta$, the MIPR has a sudden increase which characterizes a localization transition. As a comparison, we note that the localization transition does not occur for the commensurate potential system with a rational α [30], for which the wave functions of a periodic system take the Bloch form and are extended for arbitrary V.

We then perform a finite size analysis by calculating the transition points for systems with different sizes. As shown in Fig. 3, the value of the transition point $V_c(L)$ for systems with $\Delta = 0.5$ oscillates around 3.0. Defining $V_{\mathrm{avc}} = \sum_{L=L_{\min}}^{L_{\max}} V_c(L)/(L_{\max} - L_{\min})$, we calculate the average of $V_c(L)$ for different $L_{\max}$ and $L_{\min}$ and find that V_{avc} is about 3.0040 ± 0.0005 being very close to 3. The change of the gap size at $V = 2.5$ and $V = 3.5$ is shown in the inset of Fig. 3, which indicates that the gap is finite in the regime of $V < V_c$ whereas the narrow gap in the regime of $V > V_c$ approaches zero in the large L limit. We also check systems with different Δ and find similar behaviors, i.e., $V_c(L) - 2\Delta$ oscillates with L and approaches 2.0 in the large L limit.

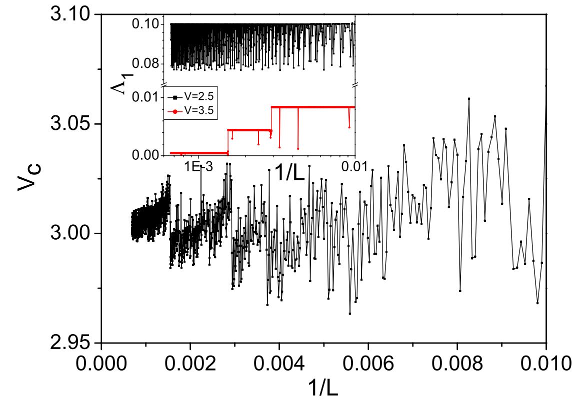

FIG. 3 (color online). The finite size analysis of the transition point, i.e., $V_c(L)$ versus $1/L$. The inset shows $\Lambda_1(L) = \Delta_g(L)/2$ versus $1/L$ in the regime of $V < V_c$ and $V > V_c$.

Next we perform an analytical derivation of the critical value V_c in the large L limit (see the Supplemental Material [25]). Rewriting the Hamiltonian [Eq. (1)] in the form of $H = \sum_{ij}[\hat{c}_i^\dagger A_{ij}\hat{c}_j + \frac{1}{2}(\hat{c}_i^\dagger B_{ij}\hat{c}_j^\dagger + \mathrm{H.c.})]$, where A is a Hermitian matrix and B is an antisymmetric matrix, we can obtain the excitation spectrum Λ_n by solving the secular equation $\det[(A + B)(A - B) - \Lambda_n^2] = 0$ [24,25]. Since the excitation gap approaches zero at the phase transition point, V_c can be determined by the condition of $\det[(A - B)(A + B)] = 0$. By using the relation $\det(A - B) = \det(A - B)^T = \det(A + B)$, we can determine V_c by $\det(A - B) = 0$, which leads to the constraint condition

$$\prod_{i=1}^{L} \cos(2\pi\alpha i) = \left(\frac{\Delta + t}{V}\right)^L \tag{5}$$

in the limit of $L \to \infty$. Taking the logarithm of the above equation and replacing the summation by an integral, we can obtain $V_c = 2(\Delta + t)e^{i2\pi n/L}$ with n being an integer. For the real solution of V_c, we have $|V_c| = 2(\Delta + t)$, which is consistent with our numerical result.

Topological features of the topological SC phase.—To characterize the topological properties of the SC phase, we seek the zero-mode solution of the system under open boundary conditions (OBC). As shown in Fig. 4(a), we plot the quasiparticle spectra of the BdG equations under OBC. In comparison with the spectra under PBC, an obvious feature is the presence of the zero-mode solution in the gap regime. The enlarged Λ_1 is shown in the inset of Fig. 4(a), which indicates a sudden increase in Λ_1 for $V > 3$. Here the zero-mode solution corresponds to the Majorana edge state with MFs localized at the ends of 1D wires. To see it clearly, we introduce the Majorana operators $\gamma_i^A = \hat{c}_i^\dagger + \hat{c}_i$ and $\gamma_i^B = (\hat{c}_i - \hat{c}_i^\dagger)/i$, which fulfill the relations $(\gamma_i^\alpha)^\dagger = \gamma_i^\alpha$ and anticommutation relations $\{\gamma_i^\alpha, \gamma_i^\beta\} = 2\delta_{ij}\delta_{\alpha\beta}$ with α and β taking A or B, and rewrite the quasiparticle operators as

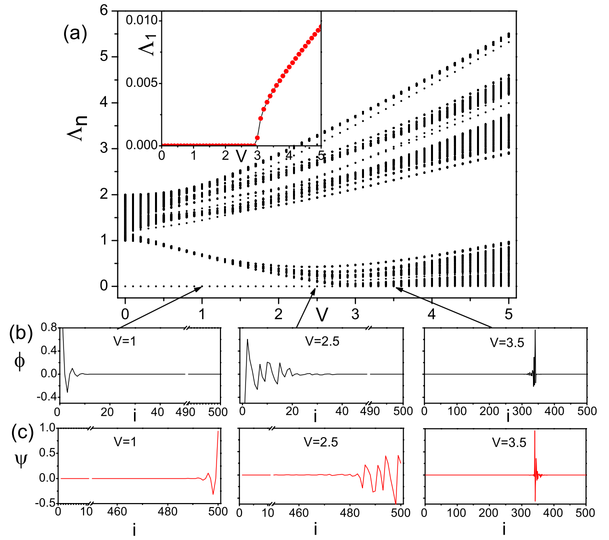

FIG. 4 (color online). (a) Energy spectra of 1D p-wave superconductors with $\alpha = (\sqrt{5}-1)/2$, $\Delta = 0.5$, and $L = 500$ under OBC. The spatial distributions of ϕ_i (b) and ψ_i (c) for the lowest excitation with various V.

$$\eta_n^\dagger = \frac{1}{2}\sum_{i=1}^{L}[\phi_{n,i}\gamma_i^A - i\psi_{n,i}\gamma_i^B], \tag{6}$$

where $\phi_{n,i} = (u_{n,i} + v_{n,i})$ and $\psi_{n,i} = (u_{n,i} - v_{n,i})$. Typical distributions of ϕ_i and ψ_i for the lowest excitation solution of Λ_1 are shown in Figs. 4(b) and 4(c). When $V < V_c$, ϕ_i (ψ_i) is located at the left (right) end and decays very quickly away from the left (right) edge. As V deviates farther from the transition point V_c, the edge mode decays more quickly. Since there is no overlap for the amplitudes of γ_i^A and γ_i^B, the zero-mode fermion splits into two spatially separated MFs. On the contrary, distributions of ϕ_i and ψ_i for the lowest excitation mode in the regime of $V > V_c$, for example $V = 3.5$, overlap together and locate inside of the bulk as a result of Anderson localization. Consequently, the corresponding quasiparticle is a localized fermion which cannot be split into two independent MFs. Therefore, the transition from TSCs to Anderson localizations can also be judged by the presence or absence of edge MFs in different parameter regimes of the system with OBC.

Z_2 topological invariant.—The existence of Majorana edge states is attributed to the nontrivial topological nature of the bulk superconductor, which can be characterized by a Z_2 topological invariant [2]. In terms of Majorana operators, the Hamiltonian [Eq. (1)] can be represented as $H = \frac{i}{4}\sum_{l,m=1}^{2L} A_{lm}\gamma_l\gamma_m$ with $A_{lm}^* = A_{lm} = -A_{ml}$, where L is the number of lattice sites, A is a skew-symmetric matrix, γ_l is defined as $\gamma_{2j-1} = \gamma_j^A$, $\gamma_{2j} = \gamma_j^B$, and $\{\gamma_l, \gamma_m\} = 2\delta_{lm}$. The nonzero matrix elements are given by $A_{2j-1,2j} = -A_{2j,2j-1} = V\cos(2\pi j\alpha)$, $A_{2j-1,2j+2} = -A_{2j+2,2j-1} = \Delta - 1$, and $A_{2j,2j+1} = -A_{2j+1,2j} = 1 + \Delta$ for $j = 1, \ldots, L$ with the boundary condition of $L + 1 = 1$. For a skew-symmetric matrix A, the Pfaffian is defined as

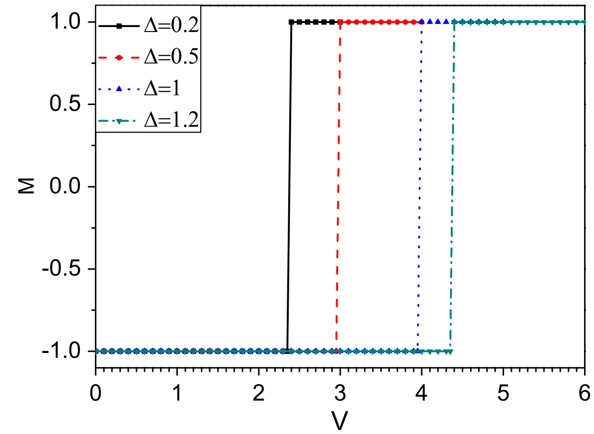

FIG. 5 (color online). Z_2 topological invariant versus V for systems with $\alpha = (\sqrt{5}-1)/2$, $L = 500$, and various Δ.

$\mathrm{Pf}(A) = \frac{1}{2^L L!}\sum_{\tau\in S_{2L}}\mathrm{sgn}(\tau)A_{\tau(1),\tau(2)}\ldots A_{\tau(2L-1),\tau(2L)}$, where S_{2L} is the set of permutations on $2L$ elements and $\mathrm{sgn}(\tau)$ is the sign of the permutation. With the Pfaffian of a system, the Z_2 topological invariant is defined as $M = \mathrm{sgn}[\mathrm{Pf}(A)]$. As shown in Fig. 5, the Z_2 topologically nontrivial phase is characterized $M = -1$, whereas the Z_2 topologically trivial phase corresponds to $M = 1$. For the system with $V < 2 + 2\Delta$, the Z_2 number $M = -1$ and the system is in the topologically nontrivial phase, while for the system with $V > 2 + 2\Delta$, the Z_2 number $M = 1$ and the system is in the topologically trivial phase. As the strength of V increases, a topological phase transition happens.

Summary.—In summary, we study the effect of disorder produced by the incommensurate potential in 1D p-wave superconductors which support a topological SC phase with Majorana edge states. Increasing the strength of disorder destroys the topological SC phase and drives the system into an Anderson localized state. The phase transition driven by the disorder is identified by analyzing the change of the gap, the long-range order of the correlation function of nonlocal operators, and the IPR which characterizes the spacial localization of wave functions. The transition point is exactly determined both numerically and analytically. A Z_2 topological invariant is also used to identify the transition from the topological SC phase, which has emergent Majorana edge states for the system with OBC, to the topologically trivial localized state.

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Note added.—Recently, we became aware of a preprint on similar topics [31].

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[1] C. W. J. Beenakker, Annu. Rev. Condens. Matter Phys. **4**, 113 (2013).

[2] A. Y. Kitaev, Phys. Usp. **44**, 131 (2001).
[3] D. A. Ivanov, Phys. Rev. Lett. **86**, 268 (2001).
[4] M. Stone and S.-B. Chung, Phys. Rev. B **73**, 014505 (2006).
[5] L. Fu and C. L. Kane, Phys. Rev. Lett. **100**, 096407 (2008).
[6] R. M. Lutchyn, J. D. Sau, and S. Das Sarma, Phys. Rev. Lett. **105**, 077001 (2010).
[7] Y. Oreg, G. Refael, and F. von Oppen, Phys. Rev. Lett. **105**, 177002 (2010).
[8] V. Mourik, K. Zuo, S. M. Frolov, S. R. Plissard, E. P. A. M. Bakkers, and L. P. Kouwenhoven, Science **336**, 1003 (2012).
[9] M. T. Deng, C. L. Yu, G. Y. Huang, M. Larsson, P. Caroff, and H. Q. Xu, Nano Lett. **12**, 6414 (2012).
[10] A. Das, Y. Ronen, Y. Most, Y. Oreg, M. Heiblum, and H. Shtrikman, Nat. Phys. **8**, 887 (2012).
[11] A. C. Potter and P. A. Lee, Phys. Rev. Lett. **105**, 227003 (2010).
[12] A. Altland and M. R. Zirnbauer, Phys. Rev. B **55**, 1142 (1997).
[13] O. Motrunich, K. Damle, and D. A. Huse, Phys. Rev. B **63**, 224204 (2001).
[14] P. W. Brouwer, A. Furusaki, I. A. Gruzberg, and C. Mudry, Phys. Rev. Lett. **85**, 1064 (2000); P. W. Brouwer, A. Furusaki, and C. Mudry, Phys. Rev. B **67**, 014530 (2003).
[15] I. A. Gruzberg, N. Read, and S. Vishveshwara, Phys. Rev. B **71**, 245124 (2005).
[16] A. M. Lobos, R. M. Lutchyn, and S. Das Sarma, Phys. Rev. Lett. **109**, 146403 (2012).
[17] P. W. Brouwer, M. Duckheim, A. Romito, and F. von Oppen, Phys. Rev. Lett. **107**, 196804 (2011); Phys. Rev. B **84**, 144526 (2011).
[18] M. Tezuka and N. Kawakami, Phys. Rev. B **85**, 140508(R) (2012).
[19] M. Tezuka and A. M. Garcia-Garcia, Phys. Rev. A **82**, 043613 (2010).
[20] S. Aubry and G. André, Ann. Isr. Phys. Soc. **3**, 133 (1980).
[21] G. Roati, C. D. Errico, L. Fallani, M. Fattori, C. Fort, M. Zaccanti, G. Modugno, M. Modugno, and M. Inguscio, Nature (London) **453**, 895 (2008).
[22] L. Jiang, T. Kitagawa, J. Alicea, A. R. Akhmerov, D. Pekker, G. Refael, J. I. Cirac, E. Demler, M. D. Lukin, and P. Zoller, Phys. Rev. Lett. **106**, 220402 (2011).
[23] P. G. de Gennes, *Superconductivity of Metals and Alloys* (Benjamin, New York, 1966).
[24] E. Lieb, T. Schultz, and D. Mattis, Ann. Phys. (N.Y.) **16**, 407 (1961).
[25] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.110.176403 for more details.
[26] D. S. Fisher, Phys. Rev. B **51**, 6411 (1995).
[27] A. P. Young and H. Rieger, Phys. Rev. B **53**, 8486 (1996).
[28] G.-L. Ingolda, A. Wobst, Ch. Aulbach, and P. Hänggi, Eur. Phys. J. B **30**, 175 (2002).
[29] D. J. Thouless, Phys. Rep. **13**, 93 (1974); M. Schreiber, J. Phys. C **18**, 2493 (1985); Y. Hashimoto, K. Niizeki, and Y. Okabe, J. Phys. A **25**, 5211 (1992).
[30] L.-J. Lang and S. Chen, Phys. Rev. B **86**, 205135 (2012).
[31] W. DeGottardi, D. Sen, and S. Vishveshwara, Phys. Rev. Lett. **110**, 146404 (2013).