# Supplementary Information for "Symmetry-Enforced Weyl Phonons"

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TABLE I: A complete list of symmetry-protected two-fold Weyl points in bosonic systems in the presence of time-reversal symmetry. The first and the second columns indicate the space group (SG) number and the corresponding high-symmetry k-point [e.g.  $\Gamma$ (GM), Z, A, M, H, K, etc.], respectively. The third and fourth columns show the abstract group (AG), which the little group of the k-point is isomorphic with, and the corresponding irreps. The fifth column indicates the Chern numbers (C) of these two-fold Weyl points. See more details for the AGs and their character tables in Ref. [1]. The asterisks indicate the high-symmetry k-points, where only the two-dimensional irrep of Weyl nodes is allowed.

SG	k	AG	irreps	C	SG	k	AG	irreps	C	SG	k	AG	irreps	C	SG	k	AG	irreps	C
* 24	W	$G_{16}^{7}$	9	$\pm 1$	75	GM	$G_4^1$	2 + 4	$\pm 2$	75	М	$G_4^1$	2 + 4	$\pm 2$	75	Ζ	$G_4^1$	2 + 4	$\pm 2$
75	Α	$G_4^1$	2 + 4	$\pm 2$	76	GM	$G_4^1$	2 + 4	$\pm 2$	76	Μ	$G_4^1$	2 + 4	$\pm 2$	77	GM	$G_4^1$	2 + 4	$\pm 2$
77	Μ	$G_4^1$	2 + 4	$\pm 2$	77	Ζ	$G_4^1$	2 + 4	$\pm 2$	77	Α	$G_4^1$	2 + 4	$\pm 2$	78	GM	$G_4^1$	2 + 4	$\pm 2$
78	Μ	$G_4^1$	2 + 4	$\pm 2$	79	GM	$G_4^1$	2 + 4	$\pm 2$	79	Z	$G_4^1$	2 + 4	$\pm 2$	79	Р	$G_2^1$	2 + 2	$\pm 2$
80	GM	$G_4^1$	2 + 4	$\pm 2$	80	Ζ	$G_4^1$	2 + 4	$\pm 2$	* 80	Р	$G_2^1$	1 + 2	$\pm 1$	89	GM	$G_8^4$	5	$\pm 2$
89	M	$G_8^4$	5	$\pm 2$	89	Ζ	$G_8^4$	5	$\pm 2$	89	A	$G_8^4$	5	$\pm 2$	90	GM	$G_{8}^{4}$	5	$\pm 2$
90	Ζ	$G_8^4$	5	$\pm 2$	91	GM	$G_8^4$	5	$\pm 2$	91	Μ	$G_8^4$	5	$\pm 2$	92	GM	$G_8^4$	5	$\pm 2$
93	GM	$G_8^4$	5	$\pm 2$	93	Μ	$G_8^4$	5	$\pm 2$	93	Z	$G_{16}^{9}$	5	$\pm 2$	93	A	$G_{16}^{9}$	5	$\pm 2$
94	GM	$G_8^4$	5	$\pm 2$	94	Ζ	$G_{16}^{9}$	5	$\pm 2$	95	GM	$G_8^4$	5	$\pm 2$	95	Μ	$G_8^4$	5	$\pm 2$
96	${\rm GM}$	$G_8^4$	5	$\pm 2$	97	${\rm GM}$	$G_8^4$	5	$\pm 2$	97	Z	$G_8^4$	5	$\pm 2$	97	Р	$G_4^2$	3 + 4	$\pm 2$
98	GM	$G_8^4$	5	$\pm 2$	98	Ζ	$G_8^4$	5	$\pm 2$	* 98	Р	$G_{16}^{7}$	10	$\pm 1$	143	GM	$G_3^1$	2 + 3	$\pm 2$
143	Α	$G_3^1$	2 + 3	$\pm 2$	144	GM	$G_3^1$	2 + 3	$\pm 2$	144	Α	$G_3^1$	2 + 3	$\pm 2$	145	GM	$G_3^1$	2 + 3	$\pm 2$
145	Α	$G_3^1$	2 + 3	$\pm 2$	146	GM	$G_3^1$	2 + 3	$\pm 2$	146	Z	$G_3^1$	2 + 3	$\pm 2$	149	GM	$G_6^2$	3	$\pm 2$
149	Α	$G_6^2$	3	$\pm 2$	150	GM	$G_6^2$	3	$\pm 2$	150	Α	$G_6^2$	3	$\pm 2$	150	K	$G_6^2$	3	$\pm 1$
150	Н	$G_6^2$	3	$\pm 1$	151	GM	$G_6^2$	3	$\pm 2$	151	Α	$G_6^2$	3	$\pm 2$	152	GM	$G_6^2$	3	$\pm 2$
152	Α	$G_6^2$	3	$\pm 2$	152	Κ	$G_6^2$	3	$\pm 1$	152	Н	$G_6^2$	3	$\pm 1$	153	GM	$G_6^2$	3	$\pm 2$
153	Α	$G_6^2$	3	$\pm 2$	154	${\rm GM}$	$G_6^2$	3	$\pm 2$	154	Α	$G_6^2$	3	$\pm 2$	154	K	$G_6^2$	3	$\pm 1$
154	Η	$G_6^2$	3	$\pm 1$	155	${\rm GM}$	$G_6^2$	3	$\pm 2$	155	Z	$G_6^2$	3	$\pm 2$	168	GM	$G_6^1$	2 + 6	$\pm 2$
168	GM	$G_6^1$	3+5	$\pm 2$	168	А	$G_6^1$	2 + 6	$\pm 2$	168	Α	$G_6^1$	3 + 5	$\pm 2$	168	K	$G_3^1$	2 + 3	$\pm 1$
168	Н	$G_3^1$	2 + 3	$\pm 1$	169	${\rm GM}$	$G_6^1$	2 + 6	$\pm 2$	169	GM	$G_6^1$	3 + 5	$\pm 2$	169	K	$G_3^1$	2 + 3	$\pm 1$
170	GM	$G_6^1$	2 + 6	$\pm 2$	170	${\rm GM}$	$G_6^1$	3 + 5	$\pm 2$	170	Κ	$G_3^1$	2 + 3	$\pm 1$	171	GM	$G_6^1$	2 + 6	$\pm 2$
171	GM	$G_6^1$	3+5	$\pm 2$	171	Α	$G_6^1$	2 + 6	$\pm 2$	171	Α	$G_6^1$	3 + 5	$\pm 2$	171	K	$G_3^1$	2 + 3	$\pm 1$
171	Н	$G_3^1$	2 + 3	$\pm 1$	172	${\rm GM}$	$G_6^1$	2 + 6	$\pm 2$	172	GM	$G_6^1$	3 + 5	$\pm 2$	172	A	$G_6^1$	2 + 6	$\pm 2$
172	Α	$G_6^1$	3 + 5	$\pm 2$	172	Κ	$G_3^1$	2 + 3	$\pm 1$	172	Н	$G_3^1$	2 + 3	$\pm 1$	173	GM	$G_6^1$	2 + 6	$\pm 2$
173	GM	$G_6^1$	3 + 5	$\pm 2$	173	Κ	$G_3^1$	2 + 3	$\pm 1$	177	GM	$G_{12}^{3}$	5	$\pm 2$	177	GM	$G_{12}^{3}$	6	$\pm 2$
177	Α	$G_{12}^{3}$	5	$\pm 2$	177	Α	$G_{12}^{3}$	6	$\pm 2$	177	Κ	$G_6^2$	3	$\pm 1$	177	Η	$G_6^2$	3	$\pm 1$
178	${\rm GM}$	$G_{12}^{3}$	5	$\pm 2$	178	${\rm GM}$	$G_{12}^{3}$	6	$\pm 2$	178	Κ	$G_6^2$	3	$\pm 1$	179	GM	$G_{12}^{3}$	5	$\pm 2$
179	${\rm GM}$	$G_{12}^{3}$	6	$\pm 2$	179	Κ	$G_6^2$	3	$\pm 1$	180	GM	$G_{12}^{3}$	5	$\pm 2$	180	GM	$G_{12}^{3}$	6	$\pm 2$
180	Α	$G_{12}^{3}$	5	$\pm 2$	180	Α	$G_{12}^{3}$	6	$\pm 2$	180	Κ	$G_6^2$	3	$\pm 1$	180	Η	$G_6^2$	3	$\pm 1$
181	GM	$G_{12}^{3}$	5	$\pm 2$	181	${\rm GM}$	$G_{12}^{3}$	6	$\pm 2$	181	Α	$G_{12}^{3}$	5	$\pm 2$	181	A	$G_{12}^{3}$	6	$\pm 2$
181	Κ	$G_6^2$	3	$\pm 1$	181	Η	$G_6^2$	3	$\pm 1$	182	GM	$G_{12}^{3}$	5	$\pm 2$	182	GM	$G_{12}^{3}$	6	$\pm 2$
182	Κ	$G_6^2$	3	$\pm 1$	195	GM	$G_{12}^{5}$	2 + 3	$\pm 4$	195	R	$G_{12}^{5}$	2 + 3	$\pm 4$	196	GM	$G_{12}^{5}$	2 + 3	$\pm 4$
196	L	$G_3^1$	2 + 3	$\pm 2$	197	GM	$G_{12}^{5}$	2 + 3	$\pm 4$	197	Н	$G_{12}^{5}$	2 + 3	$\pm 4$	198	GM	$G_{12}^5$	2 + 3	$\pm 4$
199	GM	$G_{12}^{5}$	2 + 3	$\pm 4$	199	Η	$G_{24}^{8}$	5 + 6	$\pm 4$	*199	Р	$G_{48}^{3}$	7	$\pm 1$	*199	Р	$G_{48}^{3}$	8	$\pm 1$
*199	Р	$G_{48}^{3}$	9	$\pm 1$	207	GM	$G_{24}^{7}$	3	$\pm 4$	207	Х	$G_8^4$	5	$\pm 2$	207	Μ	$G_8^4$	5	$\pm 2$
207	R	$G_{24}^{7}$	3	$\pm 4$	208	GM	$G_{24}^{7}$	3	$\pm 4$	208	Х	$G_8^4$	5	$\pm 2$	208	Μ	$G_8^4$	5	$\pm 2$
208	R	$G_{24}^{7}$	3	$\pm 4$	209	GM	$G_{24}^{7}$	3	$\pm 4$	209	Х	$G_8^4$	5	$\pm 2$	209	L	$G_6^2$	3	$\pm 2$
209	W	$G_4^2$	3 + 4	$\pm 2$	210	GM	$G_{24}^{7}$	3	$\pm 4$	210	X	$G_8^4$	5	$\pm 2$	210	L	$G_6^2$	3	$\pm 2$
*210	W	$G_{16}^{7}$	10	$\pm 1$	211	GM	$G_{24}^{7}$	3	$\pm 4$	211	Н	$G_{24}^{7}$	3	$\pm 4$	212	GM	$G_{24}^{7}$	3	$\pm 4$
213	GM	$G_{24}^{7}$	3	$\pm 4$	214	GM	$G_{24}^{7}$	3	$\pm 4$	214	Н	$G_{48}^{7}$	6	$\pm 4$	*214	Р	$G_{48}^{3}$	7	$\pm 1$
*214	Р	$G_{48}^3$	8	$\pm 1$	*214	Р	$G_{48}^{3}$	9	$\pm 1$	 									

#### A. A complete list of symmetry-protected two-fold Weyl points

In the main text, we are focused on some high-symmetry k-points, where only the two-dimensional irrep of Weyl points (WPs) is allowed. In this section, we have presented a complete list of high-symmetry k-points in Table I, where the two-dimensional irrep of WPs can be protected in bosonic systems. We call them the symmetry-protected WPs. We find that these two-fold WPs on the list can host Chern numbers of  $\pm 1$  [*i.e.*, (monopole) WPs],  $\pm 2$  [*i.e.*, double WPs] and  $\pm 4$  [*i.e.*, quadruple WPs], respectively. So far, the quadruple WPs with Chern numbers of  $\pm 4$  (*i.e.*, |C| = 4) have been rarely reported[2, 3]. Guided by the list in Table I, the quadruple WPs can be checked in future experiments.

#### B. The phonon dispersions of $K_2Sn_2O_3$ with LOTO

To illustrate that the three-fold spin-1 Weyl phonon at the high-symmetry point H is robust against the LOTO modification, the phononic dispersions of the material example  $K_2Sn_2O_3$  in the presence of the polariton of LOTO splitting are also calculated and shown in Fig. 1. The three bands including  $39^{th}$ ,  $40^{th}$  and  $41^{st}$  bands, which form the three-fold spin-1 Weyl phonon with LOTO, are also highlighted in the right panel of Fig. 1. It is clearly seen that the polariton of LOTO splitting does not split the three-fold degeneracy at H, supporting that the three-fold spin-1 Weyl phonon in the present material candidate can be observed in future experiments.



FIG. 1: The phonon dispersions of the crystal  $K_2Sn_2O_3$  with and without the polariton of LOTO splitting are drawn in the left panel, and the  $39^{th}$ ,  $40^{th}$  and  $41^{st}$  bands around the high-symmetry point H are highlighted in the right panel.

# C. The degeneracies for the $40^{th}$ and $41^{st}$ phonon bands of $K_2Sn_2O_3$

Between the  $40^{th}$  and  $41^{st}$  phonon bands of K<sub>2</sub>Sn<sub>2</sub>O<sub>3</sub> as shown in Fig.2c in the main text, there are only two spin-1 Weyl phonons at the H and  $\Gamma$  points, respectively. The Chern number of the spin-1 Weyl phonon at  $\Gamma$  is -2. To show the nontrivial surface arc states associated with them, the isofrequency surface contours at 18.145 and 18.065 THz on the (110) surface are obtained and presented in Figs. 2b and 2c, respectively. The surface arc states and the double helicoid surface states are clearly seen.



FIG. 2: **a**, The phonon dispersions on the (110) surface BZ of  $K_2Sn_2O_3$ . **b**, **c**, The surface arcs for the frequencies of 18.145 and 18.065 THz, respectively. One can see that the very long surface arcs and the double helicoid surface states appear clearly.

## D. The SEWPs at other space groups

In this section, we present some other realistic material candidates in the SGs in Table I in the main text, to illustrate the existence of the SEWPs in realistic materials. Here we investigate the material examples PNO in SG 24,  $Ag_3BiO_3$  in SG 80,  $LiAuO_2$  in SG 98,  $K_2Pb_2O_3$  in SG 199,  $SiO_2$  in SG 210 and  $K_6$  carbon in SG 214.

In the following subsections, we first draw their primitive unit cells and the first BZs to show the crystal structures, and then by using symmetry analysis and effective  $k \cdot p$  models, we verify the guiding principle of the search of SEWPs as described in the main text. Finally, we calculated their phonon dispersions and DOS, and the Chern numbers of some nontrivial phonon bands to describe the existence of the WPs. The calculation methods are the same as those described in the main text.

## 1. The SEWPs in the material PNO in SG 24

The crystallographic data of PNO are adopted from Ref. [4] and the primitive cell is shown in Fig. 3a, where the purple, gray and red atoms stand for P, N and O atoms, respectively. Each primitive cell contains 6 atoms with two P, two N and two O atoms, and the corresponding BZ is shown in Fig. 3b. PNO belongs to the body-centered orthogonal structure with the SG I2<sub>1</sub>2<sub>1</sub>2<sub>1</sub> (No. 24).

SG 24 hosts only Weyl phonons at the W point (the high-symmetry points are defined in Ref. [1], and it hosts one irreps of the AG  $G_{16}^7$  in Table I. This SG has a body-centered orthogonal Bravais lattice. The operators  $C_{2y}$  and  $C_{2z}$  acting on the primitive lattice vectors  $(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3)$  are presented [1] as follows:

$$C_{2z}\left(\mathbf{t}_{1} \ \mathbf{t}_{2} \ \mathbf{t}_{3}\right) = \left(\mathbf{t}_{1} \ \mathbf{t}_{2} \ \mathbf{t}_{3}\right) \begin{bmatrix} 0 \ 1 \ -1 \\ 1 \ 0 \ -1 \\ 0 \ 0 \ -1 \end{bmatrix}, C_{2y}\left(\mathbf{t}_{1} \ \mathbf{t}_{2} \ \mathbf{t}_{3}\right) = \left(\mathbf{t}_{1} \ \mathbf{t}_{2} \ \mathbf{t}_{3}\right) \begin{bmatrix} -1 \ 0 \ 0 \\ -1 \ 0 \ 1 \\ -1 \ 1 \ 0 \end{bmatrix}$$

Thus,

$$\begin{split} AB &= \{C_{2z} | \frac{1}{2} 0 \frac{1}{2} \} \{C_{2y} | \frac{1}{2} \frac{1}{2} 0 \} = \{C_{2x} | 1, \frac{1}{2}, \frac{1}{2} \} \\ BA &= \{C_{2y} | \frac{1}{2} \frac{1}{2} 0 \} \{C_{2z} | \frac{1}{2} 0 \frac{1}{2} \} = \{C_{2x} | 0, \frac{1}{2}, -\frac{1}{2} \} = \{E | -1, 0, -1\} AB \end{split}$$

At the W point  $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ , the pure translation operator  $\{E | -1, 0, -1\}$  is expressed as  $e^{2i\pi(-1+0-1)/4} = -1$ . Therefore, we get  $\{A, B\} = 0$ , which yields that all the phonon bands are doubly degenerate at W.

Next, we analyze a two-band model at the W point in SG 24. We have  $A^2 = \{E|000\} = 1$ ,  $B^2 = \{E|000\} = 1$  with E the identity operator and  $\{A, B\} = 0$ . With the matrix representations of  $A = \sigma_z$  and  $B = \sigma_y$ , the  $k \cdot p$ -invariant Hamiltonian is derived as (to the first order),

$$H_P^{24}(\mathbf{k}) = v_1 \sigma_x k_x + v_2 \sigma_y k_y + v_3 \sigma_z k_z$$

with  $\sigma_{x,y,z}$  Pauli matrices,  $k_{x,y,z}$  momentum offset from the P point, and  $v_{1,2,3}$  real coefficients. Obviously, it's a Weyl Hamiltonian. The P point of SG 98 and the W point of SG 210 in Table I with anti-unitary commutation relations share the similar results.

To confirm the above results from the symmetry analysis, the phononic dispersions and DOS of PNO from *ab initio* calculations are illustrated in Fig. 3c. One can see that two-fold Weyl phonons are localized at the high-symmetry point W, one of the corners of the first BZ. Note that the two linear crossing bands possess Chern numbers of  $\pm 1$ , indicating its topologically nontrivial feature.



FIG. 3: A realistic candidate PNO in SG 24. **a**, A unit cell contains 2 P, 2 N and 2 O atoms. **b**, The first BZ of PNO. **c**, The phonon dispersions of PNO along high-symmetry directions. Some nontrivial bands have Chern numbers of  $\pm 1$  around W point, which are highlighted by orange color. It is clearly seen that the two-fold Weyl phonons are localized at the high-symmetry point W, indicating that the SEWPs exist in this material.

# 2. The SEWPs in the material $Ag_3BiO_3$ in SG 80

The crystallographic data of  $Ag_3BiO_3$  are adopted from Ref. [4] and the primitive cell is shown in Fig. 4a, where the gray (rose red and red) atoms stand for Ag (Bi and O) atoms.  $Ag_3BiO_3$  belongs to the body-centered tetragonal structure with SG I4<sub>1</sub> (No. 80) [5]. A primitive cell contains 56 atoms including 24 Ag, and 8 Bi and 24 O atoms, and the corresponding BZ is shown in Fig. 4b.

SG 80 hosts only Weyl phonons at P point (the high-symmetry points are defined in Ref. [1]) with one twodimensional irrep R1R2 of the abstract group  $G_2^1$  in Table I. In  $G_2^1$ , there is only one generator  $C_{2z}$ . However, by definition it is easy to deduce that the combination operator  $\mathcal{T}C_{4z}$  can keep P invariant, although alone  $C_{4z}$  and  $\mathcal{T}$  are not symmetry operator of P, where  $\mathcal{T}$  notes the time-reversal symmetry. To prove this, we make use of the representation of  $C_{4z}$  acting on reciprocal lattice vectors ( $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$ ) in this body-centered tetragonal Bravais lattice,

$$C_{4z}\begin{pmatrix}\mathbf{g}_1\\\mathbf{g}_2\\\mathbf{g}_3\end{pmatrix} = \begin{bmatrix} 1 & 0 & -1\\ 1 & 0 & 0\\ 1 & -1 & 0 \end{bmatrix}\begin{pmatrix}\mathbf{g}_1\\\mathbf{g}_2\\\mathbf{g}_3\end{pmatrix};$$
(1)

So that at the P point  $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ ,

$$C_{4z} P = \begin{pmatrix} \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \end{pmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix} = \begin{pmatrix} \frac{3}{4}, -\frac{1}{4}, -\frac{1}{4} \end{pmatrix};$$

$$\mathcal{T}(C_{4z} P) = \begin{pmatrix} -\frac{3}{4}, \frac{1}{4}, \frac{1}{4} \end{pmatrix} = P \text{ (up to integer reciprocal lattice vectors)}$$
(2)

In this body-centered tetragonal Bravais lattice,  $C_{4z}$  acting on the primitive lattice vectors  $(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3)$  are presented in Ref. [1]:

$$C_{4z}\left(\mathbf{t}_{1} \ \mathbf{t}_{2} \ \mathbf{t}_{3}\right) = \left(\mathbf{t}_{1} \ \mathbf{t}_{2} \ \mathbf{t}_{3}\right) \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix},\tag{3}$$

Thus,

$$\mathbf{4_1}^2 = \{C_{4z} | \frac{3}{4} \frac{1}{4} \frac{1}{2} \}^2 = \{C_{2z} | 1, 0, 0\}$$
  
$$\mathbf{4_1}^4 = \{C_{4z} | \frac{3}{4} \frac{1}{4} \frac{1}{2} \}^4 = \{E | 1, 1, 0\}$$

At the P point  $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ , the pure translation operator  $\{E|1, 1, 0\}$  is expressed as  $e^{2i\pi(1+1+0)/4} = -1$ . In addition, since  $\mathcal{T}^4 = 1$ , we get  $(\mathcal{T}\mathbf{4}_1)^4 = -1$ , which yields that all the phonon bands have to be doubly degenerate at the P point. We have checked that there is no symmetry-protected degeneracy on the high-symmetry planes/lines crossing the P point.

The phononic dispersions and DOS of  $Ag_3BiO_3$  from the *ab initio* calculations are shown in Fig. 4c. It is clearly seen that two-fold SEWPs are localized at the high-symmetry point P. By fitting the two phonon bands of the lower highlighted WP of P in Fig. 4(c), the  $v_1$ ,  $v_2$  and  $v_3$  coefficients in Eq.(4) are given as 0.12, 0.12 and -1.43 THz·Å.



FIG. 4: A realistic material candidate Ag<sub>3</sub>BiO<sub>3</sub> in SG 80. **a**, A unit cell contains 56 atoms. **b**, The first BZ of Ag<sub>3</sub>BiO<sub>3</sub>. **c**, The phonon dispersions of Ag<sub>3</sub>BiO<sub>3</sub> along high-symmetry directions. It is clearly seen that the two-fold Weyl phonons are localized at the high-symmetry point P of the first BZ, and the Chern numbers  $(\pm 1)$  of the linear crossing bands are highlighted in the insets of the right panel of the figure **c**, supporting the appearance of the SEWPs in the material.

## 3. The SEWPs in the material $LiAuO_2$ in SG 98

The crystallographic data of LiAuO<sub>2</sub> are adopted from Ref. [4] and the primitive cell is shown in Fig. 5a, where the green, yellow and red atoms stand for Li, Au and O atoms, respectively. LiAuO<sub>2</sub> belongs to the body-centered tetragonal structure with SG I4<sub>1</sub>22 (No. 98). Each primitive cell contains 8 atoms with 2 Li, and 2 Au and 4 O atoms, and the corresponding BZ is shown in Fig. 5b.

SG 98 hosts only Weyl phonons at the P point (the high-symmetry points are defined in Ref. [1]), and it hosts one irrep of the AG  $G_{16}^7$  in Table I. This SG has a body-centered tetragonal Bravais lattice. The operators  $C_{2y}$  and  $C_{2z}$  acting on the primitive lattice vectors  $(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3)$  are presented [1] as follows,

$$C_{2z}\left(\mathbf{t}_{1} \ \mathbf{t}_{2} \ \mathbf{t}_{3}\right) = \left(\mathbf{t}_{1} \ \mathbf{t}_{2} \ \mathbf{t}_{3}\right) \begin{bmatrix} 0 \ 1 \ -1 \\ 1 \ 0 \ -1 \\ 0 \ 0 \ -1 \end{bmatrix};$$
$$C_{2y}\left(\mathbf{t}_{1} \ \mathbf{t}_{2} \ \mathbf{t}_{3}\right) = \left(\mathbf{t}_{1} \ \mathbf{t}_{2} \ \mathbf{t}_{3}\right) \begin{bmatrix} 0 \ -1 \ 1 \\ 0 \ -1 \ 0 \\ 1 \ -1 \ 0 \end{bmatrix},$$

Thus,

$$AB = \{C_{2z}|000\}\{C_{2y}|0\frac{1}{2}\frac{1}{2}\} = \{C_{2x}|0, -\frac{1}{2}, -\frac{1}{2}\}$$
$$BA = \{C_{2y}|0\frac{1}{2}\frac{1}{2}\}\{C_{2z}|000\} = \{C_{2x}|0, \frac{1}{2}, \frac{1}{2}\}$$
$$= \{E|0, 1, 1\}AB$$

At the P point  $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ , the pure translation operator  $\{E|0, 1, 1\}$  is expressed as  $e^{2i\pi(0+1+1)/4} = -1$ . Therefore, we get  $\{A, B\} = 0$ , which yields that all the phonon bands have to be degenerate at the P point.

The phonon dispersions and DOS of  $LiAuO_2$  from the *ab initio* calculations are shown in Fig. 5c. The two-fold SEWPs are localized at the high-symmetry point P.



FIG. 5: A realistic material candidate LiAuO<sub>2</sub> in SG 98. **a**, A unit cell contains 8 atoms. **b**, The first BZ of LiAuO<sub>2</sub>. **c**, The phonon dispersions of LiAuO<sub>2</sub> along high-symmetry directions, in which the Chern number  $(\pm 1)$  of some linear crossing bands are highlighted. It is clearly seen that the two-fold SEWPs are localized at the high-symmetry point P of the first BZ.

# 4. The SEWPs in the material $K_2Pb_2O_3$ in SG 199

The crystallographic data of  $K_2Pb_2O_3$  are adopted from Ref. [4] and the primitive cell is shown in Fig. 6a, where the purple, black and red atoms stand for K, Pb and O atoms, respectively.  $K_2Pb_2O_3$  belongs to the body-centered cubic structures with SG I2<sub>1</sub>3 (No. 199) [6]. Each primitive cell contains 6 atoms with 4 K, and 4 Pb and 6 O atoms, and the corresponding BZ is shown in Fig. 6b.

It is noted that  $K_2Pb_2O_3$  belongs to the same SG 199 which has already been studied as an example in the main text, thus the symmetry analysis and the effective  $k \cdot p$  model can be referred to the descriptions in the main text. The phonon dispersions and DOS of  $K_2Pb_2O_3$  from the *ab initio* calculations are shown in Fig. 6c. It is clearly seen that two-fold SEWPs and three-fold spin-1 SEWPs are localized at the high-symmetry points P and H, respectively.



FIG. 6: A realistic material candidate  $K_2Pb_2O_3$  in SG 199. **a**, Cubic unit cell contains 6 atoms. **b**, The first BZ of  $K_2Pb_2O_3$ . **c**, The phonon dispersions of  $K_2Pb_2O_3$  along high-symmetry directions. Similar to  $K_2Sn_2O_3$  studied in the main text, the two-fold Weyl phonons are localized at the high-symmetry point P.

# 5. The SEWPs in the material $SiO_2$ in SG 210

The crystallographic data of  $SiO_2$  are obtained from Ref. [4] and the primitive cell is shown in Fig. 7a, where the blue and red atoms stand for Li and O atoms, respectively.  $SiO_2$  belongs to the face-centered cubic structures with SG F4<sub>1</sub>32 (No. 210). Each primitive cell contains 36 atoms including 12 Si, and 24 O atoms, and the corresponding BZ is shown in Fig. 7b.

SG 210 hosts only Weyl phonons at the W point (the high-symmetry points are defined in Ref. [1]) with a twodimensional irrep R10 of the abstract group  $G_{16}^7$  in Table I. This space group is a face-centered cubic Bravais lattice. The operators  $C_{2x}$  and  $C_{2f}$  acting on the primitive lattice vectors  $(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3)$  are presented [1] as below:

$$C_{2x} \begin{pmatrix} \mathbf{t}_{1} & \mathbf{t}_{2} & \mathbf{t}_{3} \end{pmatrix} = \begin{pmatrix} \mathbf{t}_{1} & \mathbf{t}_{2} & \mathbf{t}_{3} \end{pmatrix} \begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix};$$

$$C_{2f} \begin{pmatrix} \mathbf{t}_{1} & \mathbf{t}_{2} & \mathbf{t}_{3} \end{pmatrix} = \begin{pmatrix} \mathbf{t}_{1} & \mathbf{t}_{2} & \mathbf{t}_{3} \end{pmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix},$$
(4)

Thus,

$$AB = \{C_{2x}|000\}\{C_{2f}|\frac{1}{4}\frac{1}{4}\frac{1}{4}\} = \{C_{2d}|-\frac{3}{4}\frac{1}{4}\frac{1}{4}\}$$
$$BA = \{C_{2f}|\frac{1}{4}\frac{1}{4}\frac{1}{4}\}\{C_{2x}|000\} = \{C_{2d}|\frac{1}{4}\frac{1}{4}\frac{1}{4}\}$$
$$= \{E|1,0,0\}AB$$

where  $C_{2d}$  is the Jone's symbol defined in cubic lattice as

$$C_{2d}\left(\mathbf{t}_{1} \ \mathbf{t}_{2} \ \mathbf{t}_{3}\right) \equiv \left(\mathbf{t}_{1} \ \mathbf{t}_{2} \ \mathbf{t}_{3}\right) \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix};$$
(5)

It's easy to prove that  $C_{2x}C_{2y}=C_{2y}C_{2x}=C_{2d}$ . At the W point  $(\frac{1}{2},-\frac{1}{4},\frac{3}{4})$ , the pure translation operator  $\{E|1,0,0\}$  is expressed as  $e^{2i\pi/2} = -1$ . Therefore, we get  $\{A,B\} = 0$ , which yields all the phonon bands have to be degenerate at the P point. We have checked that there is no symmetry-protected degeneracy on the high-symmetry planes/lines that cross the W point. Especially, we have to exclude those containing inversion symmetry and improper rotational symmetries.

The phonon dispersions and DOS of  $SiO_2$  from the *ab initio* calculations are shown in Fig. 7c. It is clearly seen that the two-fold SEWPs are located at the high-symmetry point W, which is in agreement with the above results from the symmetry analysis. It is noted that the phononic spectra of  $SiO_2$  in SG 210 show many imaginary frequencies, because it isn't a stable crystal.



FIG. 7: A realistic material candidate SiO<sub>2</sub> in SG 210. **a**, A cubic unit cell contains 36 atoms. **b**, The first BZ of SiO<sub>2</sub>. **c**, The phonon dispersions and DOS of SiO<sub>2</sub> along high-symmetry directions, and the Chern number  $(\pm 1)$  of some linear crossing bands are highlighted in the right panel. It is clear that the symmetry-enforced two-fold Weyl phonons are localized at the high-symmetry point W of the first BZ.

# 6. The SEWPs in the material $K_6$ carbon in SG 214

The primitive cell of  $K_6$  Carbon in SG 214 is described in Fig. 8a, where the brown atoms stand for C atoms.  $K_6$  Carbon belongs to the body-centered cubic structures with SG I4<sub>1</sub>32 (No. 214), which are in good agreements with previous calculation results [7]. Each primitive cell contains 6 C atoms, and the corresponding BZ is shown in Fig. 8b.

Similar to SG 199, SG 214 hosts only Weyl phonons at the P point (the high-symmetry points are defined in Ref. [1], even though it hosts three different two-dimensional irreps of the abstract group  $G_{48}^3$  in Table I. Since the exactly same position and symmetry of the P point in SG 214 and SG 199, we can easily get the same anti-commutation relation  $\{A, B\} = 0$  by same process where A: $\{C_{2x} | \frac{1}{2} \frac{1}{2} 0\}$ , B: $\{C_{2z} | \frac{3}{2} 1 \frac{1}{2}\}$ . The anti-commutation relation yields that all the phonon bands have to be degenerate at the P point. We have checked that there is no symmetry-protected degeneracy on the high-symmetry planes/lines that cross the P point. Especially, we have to exclude those containing inversion symmetry and improper rotation symmetries.

The phonon dispersions and DOS of  $K_6$  carbon from the *ab initio* calculations are shown in Fig. 8c. It is clearly seen that two-fold SEWPs are localized at the high-symmetry point P. By fitting the two phonon bands of the middle highlighted WP of P in Fig. 8(c), the  $v_1$ ,  $v_2$  and  $v_3$  coefficients in Eq.(3) are given as 1.66, 1.66 and 1.66 THz·Å.



FIG. 8: A realistic material candidate  $K_6$  Carbon in SG 214. **a**, A cubic unit cell contains 6 C atoms. **b**, The first BZ of  $K_6$  Carbon. **c**, The phonon dispersions of  $K_6$  Carbon along high-symmetry directions. The Chern numbers (0 and  $\pm 2$ ) of the three linear crossing bands around the high-symmetry point H, and the Chern numbers ( $\pm 1$ ) of the two linear crossing bands at the high-symmetry point P are highlighted.

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