Borromean Droplet in Three-Component Ultracold Bose Gases

Yinfeng Ma,[1,2] Cheng Peng,[1,2] and Xiaoling Cui[1,3,*]

[1]Beijing National Laboratory for Condensed Matter Physics, Institute of Physics, Chinese Academy of Sciences, Beijing 100190, China
[2]School of Physical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China
[3]Songshan Lake Materials Laboratory, Dongguan, Guangdong 523808, China

(Received 7 February 2021; revised 21 April 2021; accepted 25 June 2021; published 23 July 2021)

We investigate droplet formation in three-component ultracold bosons. In particular, we identify the formation of a Borromean droplet, where only the ternary bosons can form a self-bound droplet while any binary subsystems cannot, as the first example of Borromean binding due to a collective many-body effect. Its formation is facilitated by an additional attractive force induced by the density fluctuation of a third component, which enlarges the mean-field collapse region in comparison to the binary case and renders the formation of a Borromean droplet after incorporating the repulsive force from quantum fluctuations. Outside the Borromean regime, we demonstrate an interesting phenomenon of droplet phase separation due to the competition between ternary and binary droplets. We further show that the transition between different droplets and gas phase can be conveniently tuned by boson numbers and interaction strengths. The study reveals the rich physics of a quantum droplet in three-component boson mixtures and sheds light on the more intriguing many-body bound state formed in multicomponent systems.

DOI: 10.1103/PhysRevLett.127.043002

Introduction.—Discovering peculiar bound states helps to expand our horizons in understanding intriguing quantum effects in a physical world. Borromean binding clearly belongs to such a case, where only three items together can form the bound state while any two of them cannot. This phenomenon has been successfully reported in nuclear physics as halo nuclei [1,2] and in ultracold gases as the Efimov effect [3–5]. In these situations, the Borromean binding refers to the trimer formation in few-body clusters where no dimer is present, such as the Efimov trimers observed in the negative scattering length side [6–13] that are supported by the attractive potential due to quantum interference of three particles. Theoretical studies have found that the Borromean trimer can be equally supported by fine-tuning the shape and strength of pairwise potential [14–18] or by modifying the single-particle dispersion [19]. Given the stringent requirement for its occurrence in small clusters, whether the Borromean binding can be extended to many-body systems due to a collective effect is an interesting yet challenging problem.

On the other hand, as a typical many-body bound state, the droplet has been well studied in helium liquid [20,21] and in Bose-Einstein systems with short- and long-range interactions [22]. Recently, it has regained great attention in ultracold atoms following a pioneering proposal by Petrov [23]. Stabilized by the mean-field attraction and the Lee-Huang-Yang(LHY) repulsion from quantum fluctuations, a quantum droplet has so far been successfully observed in dipolar gases [24–30] and binary Bose gases of alkali atoms [31–34]. It has also been theoretically extended to low dimensions [35–40], Bose-Fermi mixtures [41–46], dipolar mixtures [47,48], etc.

In this Letter, we point out the first example of Borromean binding due to a collective many-body effect, namely the Borromean droplet in three-component boson mixtures. Specifically, "Borromean" means that only ternary bosons can form the droplet while any binary subsystems cannot. Its physical origin lies in an additional attractive force induced by the density fluctuation of a third component, which further intensifies the mean-field collapse and renders the formation of a Borromean droplet after incorporating the LHY repulsive force. Such a collective effect is substantially different from the mechanism of Borromean binding in small clusters [3–5,14–19]. Outside the Borromean regime, we demonstrate an interesting phenomenon of droplet phase separation due to the competition between ternary and binary droplets. The emergence of these different droplets, which is shown to be conveniently tuned by the species number and the coupling strengths, shed light on more intriguing bound state formation in multicomponent systems.

Model.—We start with the Hamiltonian for three-component bosons $H = \int d\mathbf{r} H(\mathbf{r})$, with ($\hbar = 1$):

$$H(\mathbf{r}) = \sum_{i=1,2,3} \Psi_i^\dagger(\mathbf{r}) \left(-\frac{\nabla^2}{2m_i} \right) \Psi_i(\mathbf{r}) + \sum_{ij} \frac{g_{ij}}{2} \Psi_i^\dagger \Psi_j^\dagger \Psi_j \Psi_i(\mathbf{r}). \tag{1}$$

Here $\mathbf{r}$ is the coordinate; m_i and Ψ_i are respectively the mass and field operator of boson species i; g_{ij} is the s-wave coupling strength between species i and j.

For a homogeneous system with uniform densities $\{n_i\}$ ($i = 1, 2, 3$), the mean-field energy per volume is given by

$$\epsilon_{\rm mf} = \frac{1}{2}\sum_{i,j=1}^{3} g_{ij} n_i n_j. \tag{2}$$

Following the standard Bogoliubov theory to treat quantum fluctuations [49], we obtain the LHY energy per volume as

$$\epsilon_{\rm LHY} = \int \frac{d^3\mathbf{k}}{2(2\pi)^3}\left[\sum_{i=1}^{3}(E_{i\mathbf{k}} - \epsilon_{i\mathbf{k}} - g_{ii}n_i) + \sum_{ij}\frac{2m_{ij}g_{ij}^2 n_i n_j}{\mathbf{k}^2}\right]. \tag{3}$$

Here $E_{i\mathbf{k}}$ $(i = 1, 2, 3)$ are the Bogoliubov spectra (Supplemental Material [50]).

To describe a droplet with inhomogeneous densities, we adopt the local density approximation(LDA) and write the total LHY energy as $E_{\rm LHY} = \int d\mathbf{r}\epsilon_{\rm LHY}[n_i(\mathbf{r})]$, with $n_i(\mathbf{r}) = |\Psi_i(\mathbf{r})|^2$. This leads to three coupled Gross-Pitaevskii (GP) equations for $\{\Psi_i(\mathbf{r})\}$ $(i = 1, 2, 3)$:

$$i\partial_t\Psi_i = \left[-\frac{\nabla^2}{2m_i} + \sum_j g_{ij}|\Psi_j|^2 + \frac{\partial\epsilon_{\rm LHY}}{\partial n_i}\right]\Psi_i. \tag{4}$$

The ground state can be approached by the imaginary time evolution of Eq. (4).

In this work, to facilitate discussions while keeping the essence of physics, we consider the equal mass case $m_i \equiv m$ and the coupling strengths with the following symmetry:

$$g_{11} = g_{22} \equiv g, \qquad g_{13} = g_{23} \equiv g'. \tag{5}$$

Mean-field stability.—We first analyze the mean-field stability against density fluctuations of three-component (ternary) bosons, and compare it with the two-component (binary) cases. The stability is determined by the second-order variation of $\epsilon_{\rm mf}$ with respect to a small change of local densities δn_i: $\delta^2\epsilon_{\rm mf} = \sum_{ij}\frac{1}{2}g_{ij}\delta n_i\delta n_j$, which gives

$$\delta^2\epsilon_{\rm mf} = \frac{g-g_{12}}{2}\delta n_-^2 + \frac{g+g_{12}}{2}\delta n_+^2 + \frac{g_{33}}{2}\delta n_3^2 + \sqrt{2}g'\delta n_3\delta n_+, \tag{6}$$

with $\delta n_\pm \equiv (\delta n_1 \pm \delta n_2)/\sqrt{2}$ the diagonalized fluctuation modes for components 1 and 2. Equation (6) clearly shows that the fluctuation of component 3 will interfere with δn_+ and result in two new eigenmodes, while δn_- is left unchanged. The mean-field stability requires $\delta^2\epsilon_{\rm mf} > 0$ for any δn_i, which leads to the following condition for a stable ternary system:

$$g > |g_{12}|; \qquad g_{33} > \frac{2g'^2}{g+g_{12}}. \tag{7}$$

Note that the first condition ensures the stability of the (1&2) system, while the second one is due to the interference between components 3 and (1&2) and ensures the stability of (1&2&3). Importantly, compared to the stability condition $g'^2 < gg_{33}$ for (2&3) or (1&3), the requirement in Eq. (7) is more stringent. Therefore, there exists a finite parameter window

$$\frac{g'^2}{g_{33}} \in \left(\frac{g+g_{12}}{2}, g\right), \tag{8}$$

such that all binary subsystems are stable against density fluctuations while (1&2&3) is not.

The intensified mean-field instability of ternary bosons, as compared to all binary subsystems, can be attributed to the additional attractive force brought by the third component. To see this efficiently, let us consider a special case with $g_{12} = 0$ and start from the subsystem (1&3) whose stability condition is $g'^2 < gg_{33}$. This condition can be reformulated as $g_{33} + g_{\rm ind} > 0$, with $g_{\rm ind} = -g'^2/g$ the induced interaction (to component 3) by the density fluctuation of component 1 [49]. Now if add the component 2 to (1&3), the fluctuation of component 2 will induce an additional attraction to component 3 and now $g_{\rm ind} = -2g'^2/g$ is doubled. Thus g_{33} needs to be more repulsive than in (1&3) case in order to stabilize (1&2&3). For a finite g_{12}, the fluctuations of 1 and 2 will couple together and give $g_{\rm ind} = -2g'^2/(g+g_{12})$, again more attractive than the binary cases. We have checked that such enhanced $g_{\rm ind}$ in the ternary system robustly applies for more general coupling strengths beyond component (5).

In Fig. 1, we plot out the mean-field phase diagram of (1&2&3) system in the (g', g_{12}) plane taking a fixed

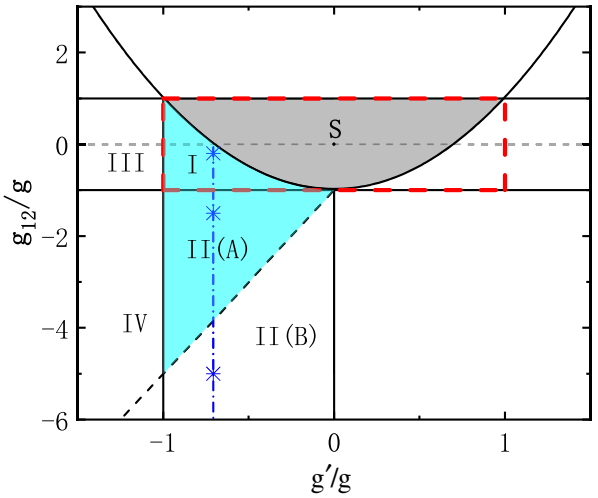

FIG. 1. Mean-field phase diagram for three-component bosons under couplings (5) and $g_{33} = g$. The gray area marks the mean-field stable region ("S"), smaller than that for binary subsystems (bounded by a red square). After incorporating the LHY repulsion, a Borromean droplet can take place in regions I and II(A). The dashed lines separating II(A) and II(B) are determined by $C_{\rm min} = 0$ at the droplet-gas transition (see text).

$g_{33} = g$. The mean-field stable region, as required by Eq. (7) and labeled "S", is shown to be smaller than the stable region of binary subsystems (bounded by a red square). For other regions in the diagram, a homogeneous (1&2&3) system will undergo a collapse or phase separation due to density fluctuations, as determined by the eigenmodes of Eq. (6) (Supplemental Material [50]). Among them, there are four regions, labeled I, II, III, IV in Fig. 1, where all three components undergo collapse simultaneously. This offers a possibility for droplet formation when further incorporating the repulsive force from quantum fluctuations. Of particular interest are regions I and II, as analyzed below.

Borromean droplet.—It is obvious that in region I, which satisfies Eq. (8), the ternary system (1&2&3) can form a self-bound droplet while any binary subsystem cannot. By definition, this is the *Borromean droplet*. Meanwhile, because the actual droplet formation also depends on particle numbers $\{N_i\}$, it is possible for such an intriguing state to exist in other regions by properly tuning $\{N_i\}$. Given the coupling symmetry in Eq. (5), we have $N_1 = N_2$ for the ground state and there left two tunable parameters for $\{N_i\}$: total number $N = 2N_1 + N_3$ and number ratio $C = N_3/N_1$.

To explore the essential properties of the Borromean droplet, we carry out full simulations of the GP equations (4) to search for ground state with a fixed $g'/g = -\sqrt{2}/2$ and different g_{12}/g in regions I and II, i.e., following the vertical line in Fig. 1. In Figs. 2(a)–2(c), we show the area of droplet formation in the $N - C$ plane for three typical values of g_{12}/g, where we also show the value of C when the energy reaches a minimum for each N, denoted as $C_{\min}$ (dashed lines in Fig. 2). One can see that for g_{12}/g in region I [Fig. 2(a)], a ternary droplet can be supported when N is beyond a critical number, $N_{t,c}$, where a gas to droplet transition occurs. For all $N > N_{t,c}$, the droplet can survive for C only within a narrow window around $C_{\min}$. This characterizes the Borromean nature of the droplet; i.e., its formation cannot extend to $C = 0$ (when the third component is absent).

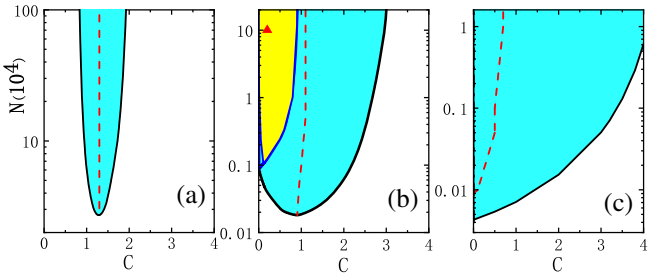

FIG. 2. Droplet region (colored) in the $N - C$ plane for three typical points (marked by *) on the vertical line in Fig. 1, which have a fixed $g'/g = -\sqrt{2}/2$ and different $g_{12}/g = -0.2(a); -1.5(b); -5(c)$. Red dashed lines show $C_{\min}$ when the system has minimal total energy at given N. In (b), the yellow area marks the regime for droplet phase separation.

Interestingly, the Borromean droplet can extend to part of region II. As an example, for the parameters considered in Fig. 2(b), we can see that as N increases, a ternary droplet first emerges at $N_{t,c}$ with a finite C. As N is further increased to $N_{b,c}$, the binary (1&2) droplet appears at $C = 0$. The Borromean droplet is then stabilized within the number window $N \in (N_{t,c}, N_{b,c})$, where N is large enough to support a ternary droplet but still small for the binary one. The Borromean droplet will vanish when go deep into region II. As shown in Fig. 2(c) for large attractive g_{12}, as N increases the droplet solution first emerges at $N_{b,c}$ with $C = 0$. In this case, a binary droplet is more favored than a ternary one.

Given the different behaviors of droplet formation in region II, we have separated this region into II(A) and II(B) in Fig. 1: the former can support the Borromean droplet (within a certain number window) while the latter cannot. Their boundary (dashed line in Fig. 1) is determined by the zero crossing of $C_{\min}$ at critical $N_{t,c}$. In Fig. 3(a), we show $C_{\min}$ at $N_{t,c}$ as a function of g_{12}/g with a given $g'/g = -\sqrt{2}/2$. We can see that $C_{\min}$ continuously decreases as g_{12} becomes more attractive and decreases to zero at $g_{12}/g \approx -4$, which separates region II(A) from II(B)

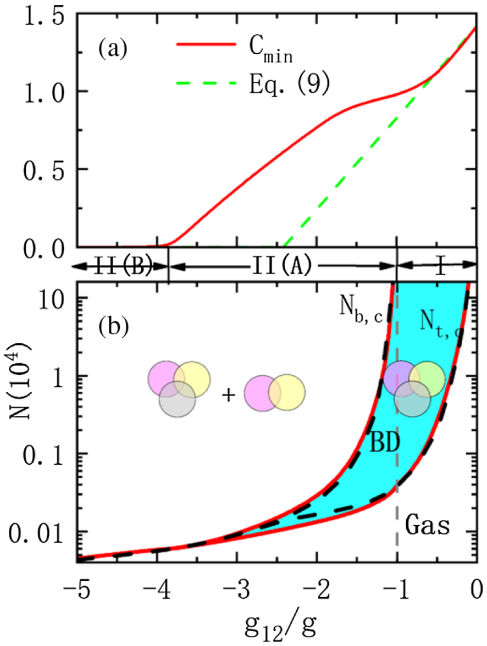

FIG. 3. (a) $C_{\min}$ (solid line) as a function of g_{12}/g at the critical number $N_{t,c}$, in comparison with $C_{\min}^{(0)}$ (dashed line) from Eq. (9). (b) Phase diagram in the $N - g_{12}$ plane. The Borromean droplet ("BD") occurs in region I for $N > N_{t,c}$ and in II(A) for $N \in (N_{t,c}, N_{b,c})$. For $N < N_{t,c}$ the system is in the gas phase; for $N > N_{b,c}$ the binary droplet can also exist. Dashed lines show the function fits of $N_{t,c}^{(0)}$ [Eq. (11)] and $N_{b,c}^{(0)}$ (from Ref. [23]). Here $g'/g = -\sqrt{2}/2$.

on the vertical line in Fig. 1. In fact, $C_{\min}$ can be estimated from the minimization of $\epsilon_{\rm mf}$, which gives

$$C^{(0)}_{\min} = \frac{g + g_{12} - 2g'}{g - g'}. \tag{9}$$

In Fig. 3(a), we can see that Eq. (9) well fits $C_{\min}$ in region I but deviates visibly as entering region II(A) when the system is far away from the mean-field collapse line.

In Fig. 3(b), we further map out the phase diagram highlighting the Borromean droplet(BD) in the (g_{12}, N) plane, taking a fixed $g'/g = -\sqrt{2}/2$. To summarize, the Borromean droplet occurs at $N > N_{t,c}$ in region I and $N \in (N_{t,c}, N_{b,c})$ in region II(A). We can see that both $N_{t,c}$ and $N_{b,c}$ decrease as g_{12} gets more attractive.

Now we analytically estimate $N_{t,c}$. First, we investigate the equilibrium density of the Borromean droplet by enforcing the zero pressure $P = \sum_i n_i \partial\epsilon/\partial n_i - \epsilon = 0$, where $\epsilon = \epsilon_{\rm mf} + \epsilon_{\rm LHY}$. Utilizing $\epsilon_{\rm mf} = g n_1^2 f_1$ with $f_1 = C^2/2 + 2Cg'/g + 1 + g_{12}/g$, and the LHY energy at the mean-field collapse line (Supplemental Material [50]) $\epsilon_{\rm LHY} = 8/(15\pi^2)(g n_1)^{5/2} f_2$, with $f_2 = (1 + g_{12}/g + C)^{5/2} + (1 - g_{12}/g)^{5/2}$, we obtain the density of component i in the ternary droplet

$$n^{(0)}_{t,i} = \eta_i \frac{25\pi}{1024 a^3} \left(\frac{f_1}{f_2}\right)^2. \tag{10}$$

Here $a = mg/(4\pi)$, $\eta_1 = \eta_2 = 1$, and $\eta_3 = C^{(0)}_{\min}$. Further, based on Eqs. (9) and (10) and the single-mode assumption $\Psi_i(\mathbf{r}) = \sqrt{n^{(0)}_{t,i}}\psi(\mathbf{r})$, the coupled GP equations (4) can be reduced to a single one similar to that in the binary case [23]. This results in the following critical number at the transition between the ternary droplet and the gas phase:

$$N^{(0)}_{t,c} = (2 + C)^{5/2} \left(\frac{3}{2}\right)^{3/2} \frac{4\tilde{N}_c}{5\pi^2} \frac{f_2}{f_1^2}, \tag{11}$$

with $\tilde{N}_c = 18.65$ (at the vanishing of droplet solution). In Fig. 2(c), we show that (11) fits numerical $N_{t,c}$ qualitatively well over the parameter regime considered. When $C = 0$, Eqs. (10) and (11) recover the results for binary droplets [23].

Droplet phase separation.—Outside the Borromean regime, both the ternary and binary droplets can survive and will directly compete with each other. Here we will demonstrate an interesting phenomenon of droplet phase separation.

We consider the region II(A) with a large $N(> N_{b,c})$, i.e., above the BD region in Fig. 3(b). In this case, as shown in Fig. 1(b), the droplet solution can appear in a reasonably broad range of C and its energy minimum occurs at $C_{\min} \neq 0$. Among these C values, $C = 0$ and $C = C_{\min}$ represent two typical solutions corresponding to, respectively, the binary(1&2) and ternary(1&2&3) droplets. We find that for certain intermediate C, two types of droplet can coexist in the form of phase separation. As shown in Fig. 4, it is manifested by two different plateaus in the density profile that well fit the equilibrium densities of ternary and binary droplets. Specifically, droplet (1&2&3) occupies the center with $C \sim C_{\min}$ and a higher density, while droplet (1&2) stays at edge with $C = 0$ and a lower density. Such a distribution is believed to lower the surface energy the most.

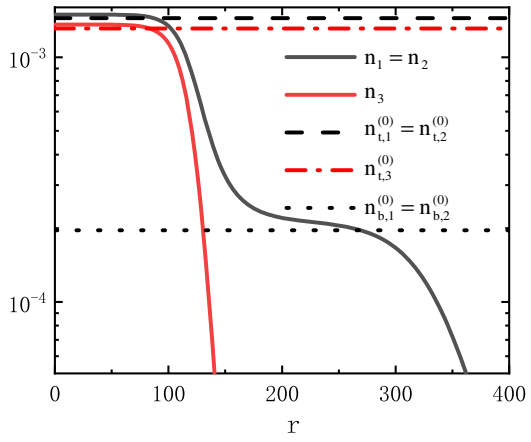

FIG. 4. Density profile displaying the phase separation between ternary (1&2&3) and binary (1&2) droplets. Here $g'/g = -\sqrt{2}/2$, $g_{12}/g = -1.5$, $N = 10^5$, $C = 0.2$, corresponding to the triangular point in Fig. 2(b). Horizontal lines show function fits to the equilibrium densities of ternary [Eq. (10)] and binary (from Ref. [23]) droplets. The length and density units are, respectively, a and $1/a^3$ $[a = mg/(4\pi)]$.

Here we estimate the (N, C) parameter regime to support the droplet phase separation. For given N and C, we have $N_3 = NC/(C + 2)$ and $N_1 = N_2 = N/(C + 2)$. Since the full number of component 3, together with part of (1&2) components, occupy at the center to form droplet (1&2&3) with number ratio $C_{\min}$, the total number of ternary droplet is $N_t = N_3(C_{\min} + 2)/C_{\min}$. The remaining components 1 and 2 are left to form the binary droplet with number $N_b = 2N_1 - 2N_3/C_{\min}$. The appearance of two types of droplets thus requires $N_t > N_{t,c}$ and $N_b > N_{b,c}$, setting the constraint for allowed $N - C$ values. For any given $N(> N_{b,c})$, the allowed C is within a certain window (C_L, C_H), as shown by yellow region in Fig. 2(b). For very large N, we have $C_L \sim 0$ and $C_H \sim C_{\min}$, and thus the droplet phase separation can occur for any $C \in (0, C_{\min})$.

Summary and discussion.—In summary, we have shown that droplet formation in three-component bosons can exhibit much richer physics than that in binary systems, including the enhanced density fluctuations toward mean-field collapse, the occurrence of the Borromean droplet, as well as the competition and phase separation between different types of droplets. Though we have focused on

the equal mass case with certain coupling symmetries (5), the underlying physics revealed here is robust and can be extended to the more general case of mass ratios and coupling strengths.

For the experimental detection of our results, especially regions I and II(A) in Fig. 1, one would need the three-component bosons to hold (i) all repulsive intraspecies couplings and (ii) at least two interspecies couplings to be attractive. A good starting point is to first find $|1\rangle$ and $|2\rangle$ that may support a binary droplet, such as two $|F=1\rangle$ hyperfine states $|m_F=0\rangle$ and $|m_F=-1\rangle$ of the ^{39}K atom [51,52] used to observe quantum droplets near $B_0 \sim 57$G [31–33]. Then a third component $|3\rangle$ is required to own a repulsive coupling itself and interact attractively with $|1\rangle$ or $|2\rangle$ near B_0. Key observations would include the emergence of a ternary droplet without any binary ones, and the spontaneous phase separation when they coexist. As more and more Feshbach resonances are explored between heteronuclear bosons, such as ^{41}K-^{87}Rb [53], ^{39}K-^{87}Rb [54], ^{23}Na-^{87}Rb [55], ^{39}K-^{133}Cs [56], etc., it would be promising in the future to find the proper three-component mixtures and detect the droplet physics therein.

Finally, we discuss the possibility of droplet formation with high-order Borromean structure, i.e., following the Brunnian ring [57–59]. We remark that it is possible to form the nth order Brunnian droplet, where only the n-component bosons together can form a self-bound state while any $m(<n)$. component cannot. The underlying mechanism resembles that of the Borromean droplet revealed in this work; i.e., an extra component brings an additional attractive force to the system via density fluctuations. Consider a simple case in which $g_{ii}=g$ $(i=2,\ldots n)$ and all zero interspecies couplings except $g_{1i}=g'$, the nth order Brunnian droplet will occur if $gg_{11}/g'^2 \in (n-2, n-1)$. This shows the power of a collective many-body effect in engineering bound states with a diversely fascinating structure.

This work is supported by the National Key Research and Development Program of China (Grants No. 2016YFA0300603 and No. 2018YFA0307600), the National Natural Science Foundation of China (Grant No. 12074419), and the Strategic Priority Research Program of Chinese Academy of Sciences (Grant No. XDB33000000).

*xlcui@iphy.ac.cn

[1] M. V. Zhukov, B. V. Danilin, D. V. Fedorov, J. M. Bang, I. S. Thompson, and J. S. Vaagen, Phys. Rep. **231**, 151 (1993).
[2] D. V. Fedorov, A. S. Jensen, and K. Riisager, Phys. Rev. C **49**, 201 (1994).
[3] E. Braaten and H.-W. Hammer, Phys. Rep. **428**, 259 (2006).
[4] C. H. Greene, P. Giannakeas, and J. Pérez-Ríos, Rev. Mod. Phys. **89**, 035006 (2017).
[5] P. Naidon and S. Endo, Rep. Prog. Phys. **80**, 056001 (2017).
[6] T. Kraemer, M. Mark, P. Waldburger, J. G. Danzl, C. Chin, B. Engeser, A. D. Lange, K. Pilch, A. Jaakkola, H.-C. Nägerl, and R. Grimm, Nature (London) **440**, 315 (2006).
[7] T. B. Ottenstein, T. Lompe, M. Kohnen, A. N. Wenz, and S. Jochim, Phys. Rev. Lett. **101**, 203202 (2008).
[8] J. R. Williams, E. L. Hazlett, J. H. Huckans, R. W. Stites, Y. Zhang, and K. M. O'Hara, Phys. Rev. Lett. **103**, 130404 (2009).
[9] M. Zaccanti, B. Deissler, C. D'Errico, M. Fattori, M. Jona-Lasinio, S. Müller, G. Roati, M. Inguscio, and G. Modugno, Nat. Phys. **5**, 586 (2009).
[10] N. Gross, Z. Shotan, S. Kokkelmans, and L. Khaykovich, Phys. Rev. Lett. **103**, 163202 (2009).
[11] S. E. Plooack, D. Dries, and R. G. Hulet, Science **326**, 1683 (2009).
[12] M. Berninger, A. Zenesini, B. Huang, W. Harm, H.-C. Nägerl, F. Ferlaino, R. Grimm, P. S. Julienne, and J. M. Hutson, Phys. Rev. Lett. **107**, 120401 (2011).
[13] R. J. Wild, P. Makotyn, J. M. Pino, E. A. Cornell, and D. S. Jin, Phys. Rev. Lett. **108**, 145305 (2012).
[14] J.-M. Richard and S. Fleck, Phys. Rev. Lett. **73**, 1464 (1994).
[15] S. Moszkowski, S. Fleck, A. Krikeb, L. Theussl, J. M. Richard, and K. Varga, Phys. Rev. A **62**, 032504 (2000).
[16] E. Nielsen, D. V. Fedorov, and A. S. Jensen, Few-Body Syst. **27**, 15 (1999).
[17] A. G. Volosniev, D. V. Fedorov, A. S. Jensen, and N. T. Zinner, Eur. Phys. J. D **67**, 95 (2013).
[18] A. G. Volosniev, D. V. Fedorov, A. S. Jensen, and N. T. Zinner, J. Phys. B **47**, 185302 (2014).
[19] X. Cui and W. Yi, Phys. Rev. X **4**, 031026 (2014).
[20] S. Grebenev, J. P. Toennies, and A. F. Vilesov, Science **279**, 2083 (1998).
[21] M. Barranco, R. Guardiola, S. Hernandez, R. Mayol, and J. Navarro, J. Low Temp. Phys. **142**, 1 (2006).
[22] K. Huang, Phys. Rev. **115**, 765 (1959); **119**, 1129 (1960).
[23] D. S. Petrov, Phys. Rev. Lett. **115**, 155302 (2015).
[24] I. Ferrier-Barbut, H. Kadau, M. Schmitt, M. Wenzel, and T. Pfau, Phys. Rev. Lett. **116**, 215301 (2016).
[25] M. Schmitt, M. Wenzel, F. Böttcher, I. Ferrier-Barbut, and T. Pfau, Nature (London) **539**, 259 (2016).
[26] I. Ferrier-Barbut, M. Schmitt, M. Wenzel, H. Kadau, and T. Pfau, J. Phys. B **49**, 214004 (2016).
[27] L. Chomaz, S. Baier, D. Petter, M. J. Mark, F. Wächtler, L. Santos, and F. Ferlaino, Phys. Rev. X **6**, 041039 (2016).
[28] L. Tanzi, E. Lucioni, F. Fama, J. Catani, A. Fioretti, C. Gabbanini, R. N. Bisset, L. Santos, and G. Modugno, Phys. Rev. Lett. **122**, 130405 (2019).
[29] F. Böttcher, J.-N. Schmidt, M. Wenzel, J. Hertkorn, M. Guo, T. Langen, and T. Pfau, Phys. Rev. X **9**, 011051 (2019).
[30] L. Chomaz, D. Petter, P. Ilzhöfer, G. Natale, A. Trautmann, C. Politi, G. Durastante, R. M. W. van Bijnen, A. Patscheider, M. Sohmen, M. J. Mark, and F. Ferlaino, Phys. Rev. X **9**, 021012 (2019).
[31] C. R. Cabrera, L. Tanzi, J. Sanz, B. Naylor, P. Thomas, P. Cheiney, and L. Tarruell, Science **359**, 301 (2018).
[32] P. Cheiney, C. R. Cabrera, J. Sanz, B. Naylor, L. Tanzi, and L. Tarruell, Phys. Rev. Lett. **120**, 135301 (2018).

[33] G. Semeghini, G. Ferioli, L. Masi, C. Mazzinghi, L. Wolswijk, F. Minardi, M. Modugno, G. Modugno, M. Inguscio, and M. Fattori, Phys. Rev. Lett. **120**, 235301 (2018).
[34] C. D'Errico, A. Burchianti, M. Prevedelli, L. Salasnich, F. Ancilotto, M. Modugno, F. Minardi, and C. Fort, Phys. Rev. Research **1**, 033155 (2019).
[35] D. S. Petrov and G. E. Astrakharchik, Phys. Rev. Lett. **117**, 100401 (2016).
[36] D. Edler, C. Mishra, F. Wächtler, R. Nath, S. Sinha, and L. Santos, Phys. Rev. Lett. **119**, 050403 (2017).
[37] K. Jachymski and R. Oldziejewski, Phys. Rev. A **98**, 043601 (2018).
[38] P. Zin, M. Pylak, T. Wasak, M. Gajda, and Z. Idziaszek, Phys. Rev. A **98**, 051603(R) (2018).
[39] T. Ilg, J. Kumlin, L. Santos, D. S. Petrov, and H. P. Büchler, Phys. Rev. A **98**, 051604(R) (2018).
[40] X. Cui and Y. Ma, Phys. Rev. Research **3**, L012027 (2021).
[41] X. Cui, Phys. Rev. A **98**, 023630 (2018).
[42] S. Adhikari, Laser Phys. Lett. **15**, 095501 (2018).
[43] D. Rakshit, T. Karpiuk, M. Brewczyk, and M. Gajda, SciPost Phys. **6**, 079 (2019).
[44] D. Rakshit, T. Karpiuk, P. Zin, M. Brewczyk, M. Lewenstein, and M. Gajda, New J. Phys. **21**, 073027 (2019).
[45] M. Wenzel, T. Pfau, and I. Ferrier-Barbut, Phys. Scr. **93**, 104004 (2018).
[46] J.-B. Wang, J.-S. Pan, X. Cui, and W. Yi, Chin. Phys. Lett. **37**, 076701 (2020).
[47] J. C. Smith, D. Baillie, and P. B. Blakie, Phys. Rev. Lett. **126**, 025302 (2021).
[48] R. N. Bisset and L. A. Peña Ardila, and L. Santos, Phys. Rev. Lett. **126**, 025301 (2021).
[49] C. J. Pethick and H. Smith, *Bose-Einstein Condensation in Dilute Gases* (Cambridge University Press, Cambridge, England, 2002).
[50] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.127.043002 for the more details on the derivation of mean-field instability and the LHY energy.
[51] M. Lysebo and L. Veseth, Phys. Rev. A **81**, 032702 (2010).
[52] S. Roy, M. Landini, A. Trenkwalder, G. Semeghini, G. Spagnolli, A. Simoni, M. Fattori, M. Inguscio, and G. Modugno, Phys. Rev. Lett. **111**, 053202 (2013).
[53] G. Thalhammer, G. Barontini, L. De Sarlo, J. Catani, F. Minardi, and M. Inguscio, Phys. Rev. Lett. **100**, 210402 (2008).
[54] L. Wacker, N. B. Jørgensen, D. Birkmose, R. Horchani, W. Ertmer, C. Klempt, N. Winter, J. Sherson, and J. J. Arlt, Phys. Rev. A **92**, 053602 (2015).
[55] F. Wang, X. Li, D. Xiong, and D. Wang, J. Phys. B **49**, 015302 (2016).
[56] M. Gröbner, P. Weinmann, E. Kirilov, H.-C. Nägerl, P. S. Julienne, C. Ruth Le Sueur, and J. M. Hutson, Phys. Rev. A **95**, 022715 (2017).
[57] H. Brunn, Math. Phys. Klasse **22**, 77 (1892).
[58] H. Debrunner, Duke Math. J. **28**, 17 (1961); D. E. Penney, Duke Math. J.**36**, 31 (1969); T. Yanagawa, Osaka J. Math. **1**, 127 (1964), https://projecteuclid.org/journals/osaka-journal-of-mathematics/volume-1/issue-2/Brunnian-systems-of-2-spheres-in-4-space/ojm/1200691078.full.
[59] N. A. Baas, D. V. Fedorov, A. S. Jensen, K. Riisager, A. G. Volosniev, and N. T. Zinner, Phys. At. Nucl. **77**, 336 (2014).