Non-Abelian Josephson Effect between Two $F = 2$ Spinor Bose-Einstein Condensates in Double Optical Traps

Ran Qi,[1] Xiao-Lu Yu,[1,2] Z. B. Li,[2] and W. M. Liu[1]

[1]*Beijing National Laboratory for Condensed Matter Physics, Institute of Physics, Chinese Academy of Sciences, Beijing 100080, China*
[2]*School of Physics and Engineering, Sun Yat-Sen University, Guangzhou 510275, China*
(Received 24 September 2008; revised manuscript received 22 March 2009; published 4 May 2009)

We investigate the non-Abelian Josephson effect in $F = 2$ spinor Bose-Einstein condensates with double optical traps. We propose a real physical system which contains non-Abelian Josephson effect and has very different density and spin tunneling characters compared with the Abelian case. We calculate the frequencies of the pseudo Goldstone modes in different phases between two traps, respectively, which are the crucial feature of the non-Abelian Josephson effect. We also give an experimental protocol to observe this novel effect in future experiments.

Introduction.—The Josephson effect is a quantum tunneling phenomenon which occurs when a pair of superconducting or superfluid systems are weakly linked by some kind of physical barrier. Beginning with Josephson's original paper in 1962 [1], the Josephson effect has become a paradigm of the phase coherence manifestation in a macroscopic quantum system. With the rapid experimental progress in cold atom physics, the Josephson junction has been realized for the trapped Bose-Einstein condensates (BEC) of ^{87}Rb [2] and ^{23}Na [3]. However, most of the extensive studies about this effect focus on the Abelian case so far, in terms of a junction of two systems with spontaneously broken Abelian symmetry [4,5]. There are also some kinds of Josephson-type effect without junctions, such as the spin mixing process of spin-1 condensate in single well discussed in Ref. [6].

Recently, Esposito *et al.* generalized the Abelian Josephson effect to the non-Abelian case in a field theoretic language [7]. The non-Abelian Josephson effect emerges in a junction of two weakly coupled systems with spontaneously broken non-Abelian symmetries, which often involves multicomponent order parameters. The non-Abelian nature of the symmetry will induce more than one kind of tunneling mode in the Josephson effect. These different tunneling modes can be characterized by the excitation of so-called pseudo Goldstone bosons which have small but finite masses [7]. The emergence of pseudo Goldstone bosons is a consequence of the symmetry breaking term due to the coupling between the two condensates. Theoretically speaking, the non-Abelian Josephson effect is ubiquitous in nature, covering many topics from particle physics to condensed matter physics. For example, it may be realized in the superfluid ^{3}He Josephson weak link [8], high density phases of QCD [9], and artificial non-Abelian gauge field induced by nonlinear optics [10]. However, there are no specific experimental constructions so far.

How can an experimental protocol be designed to observe this novel effect in future experiments? To our knowledge, this effect has not been explicitly spelled out in any real physical system, which is what we attempt to do here. In order to generalize to the non-Abelian junction in experiments, we need a system of multicomponent order parameter which has a non-Abelian symmetry in the ground state. In contrast with magnetic trap, the spin of the alkali atoms is essentially free in an optical trap [11–13]. This spinor nature properly provides the scenario of our non-Abelian construction. We now briefly introduce the system about a spinor atomic BEC in a double-well optical trap. Although the dynamical tunneling properties of spin-1 and ''pseudo spin-1/2'' bosonic systems have been calculated [14–18], the essence of the non-Abelian effect has not been captured yet. In present Letter, we focus on the pseudo Goldstone modes due to the non-Abelian symmetry breaking, which is at the heart of Josephson effect. For concrete construction, we propose spin-2 BEC in double optical traps. The spin-2 system has possible advantages, compared with the spin-1 system, in the sense that the symmetry properties are much richer to explore the non-Abelian effect. We should note that in the spin-1 system some elements of the symmetry group are hidden in the ground state. For example, in the polar state of the spin-1 system, the symmetry group of ground state is $U(1) \times S(2)$ which is a subgroup of the symmetry group of the Hamiltonian $U(1) \times SO(3)$ [13]. The reason why this happens is that in the low spin system, some rotation in the symmetry group of the Hamiltonian will leave the ground state unchanged and does not contribute to Goldstone modes. In contrast, in the antiferromagnetic and cyclic phase of spin-2 condensate, the full $U(1) \times SO(3)$ symmetry is preserved in the ground state configuration, except for some specific values of the parameter [19].

Ground state structure.—Let us consider a system of a homogenous spin-2 Bose gas with *s*-wave interaction. This

system can be described by the following mean field free energy:

$$F(\psi) = \frac{1}{2}\left[c_0(\psi^\dagger\psi)^2 + c_1\langle\mathbf{f}\rangle^2 + \frac{c_2}{5}|\Theta|^2 \right] - \mu\psi^\dagger\psi, \quad (1)$$

where $\psi = (\psi_2, \psi_1, \psi_0, \psi_{-1}, \psi_{-2})^T$ is the order parameter of the system, c_0, c_1, and c_2 are interaction strengths related to the scattering length in different spin channels, $\langle\mathbf{f}\rangle = \psi^\dagger\mathbf{f}\psi$ is the mean value of the spin operator, and $\Theta = \sum_{a=-2}^{2}(-1)^a\psi_a\psi_{-a}$ represents a single pair of identical spin-2 particles. The ground state configuration can be determined by minimizing this free energy. There are several distinct phases in this system [20,21]. We will discuss these phases under zero magnetic field and analyze the symmetry and low-lying excitation spectrum of each phase. (I) Ferromagnetic phases: When $c_1 < 0$ and $c_1 - c_2/20 < 0$, two kinds of ferromagnetic phases are energetically favored. The corresponding ground state configurations are given by $\psi = \sqrt{n}e^{i\theta}(1, 0, 0, 0, 0)$ or $\psi = \sqrt{n}e^{i\theta}(0, 1, 0, 0, 0)$, where $n = \mu/(c_0 + 4c_1)$ is the particle density and θ is an arbitrary global phase. It is obvious that these ground states have a $U(1)$ symmetry which leads to only one massless Goldstone mode. Therefore, two uncoupled systems have a $U(1)\bigotimes U(1)$ symmetry. This symmetry will softly break into a $U(1)$ diagonal symmetry when a weak coupling is applied. This pattern of symmetry breaking corresponds to an Abelian Josephson effect. The low-lying excitation spectrum of this state has been derived as $\omega_k = \sqrt{\epsilon_k(\epsilon_k + 2g_4n)}$ [20], where $\epsilon_k = k^2$ and $g_4 = c_0 + 4c_1$. We should note that this Goldstone mode will break into two modes when coupling is applied: one zero energy mode and one pseudo Goldstone mode. This pseudo Goldstone mode has a finite but small gap and leads to a density mode fluctuation in dc Josephson current. (II) Antiferromagnetic phase: In the absence of magnetic field, there is only one kind of antiferromagnetic phase when $c_2 < 0$ and $c_1 - c_2/20 > 0$ are satisfied. The corresponding ground state configuration is degenerate with respect to five continuous variables which lead to five massless Goldstone modes [21]. Four of them correspond to the $U(1) \times SO(3)$ symmetry of the free energy. The extra degeneracy besides the $U(1) \times SO(3)$ symmetry is correct only on the mean field level and will be removed when the quantum fluctuation is considered. In this Letter, we will work on the mean field level and maintain all the five Goldstone modes. The effect of quantum fluctuation will be included in future work. (III) Cyclic phase: When $c_1 > 0$ and $c_2 > 0$, the cyclic phase is energetically favored. The ground state configuration is given by $\psi = \sqrt{n}(\frac{e^{i\theta_2}}{2}, 0, \frac{e^{i\theta_0}}{\sqrt{2}}, 0, \frac{e^{i\theta_{-2}}}{2})$ where $n = \mu/c_0$ is the particle density and the global phase $\theta_{\pm 2}$ and θ_0 satisfy $\theta_2 + \theta_{-2} - 2\theta_0 = \pi$ [21,22]. Using the Schwinger boson representation [19], one can see that the ground state of cyclic phase is mapped to a tetrahedron on the unit sphere. Therefore, the ground state of cyclic phase has a full $U(1) \times SO(3)$ degeneracy and leads to four Goldstone modes. We will

show that these modes also lead to non-Abelian Josephson effect.

Non-Abelian Josephson effect.—We will analyze the Josephson effect of a spin-2 BEC system in a double-well optical trap, as shown in Fig. 1. We assume that the energy barrier between the two wells is strong enough so that the coupling between the Bose gas in each well is very weak and the overlap of the ground state wave functions in left and right well [which we denote as $\varphi_L(\mathbf{x})$ and $\varphi_R(\mathbf{x})$] can be safely neglected. We will also use the single mode approximation which means we take the same mode function for all five spin components; this is a widely used approximation and it is valid when the spin interaction is symmetric. Under these assumptions, the system can be described by the following potential:

$$V_{\text{couple}} = F(\psi_L) + F(\psi_R) - J(\psi_L^\dagger\psi_R + \psi_R^\dagger\psi_L), \quad (2)$$

where ψ_L and ψ_R are the order parameter of the Bose system in the left and right well, respectively, and J is the coupling parameter. It should be noted that we have taken the same chemical potential for the Bose gas in the right and left well, because we will be interested in only the dc Josephson effect which captures the essence of the non-Abelian symmetry breaking as simple as possible. Using the dynamic equations of the condensate $i\frac{d}{dt}\psi_{aL} = \frac{\delta V_{\text{couple}}}{\delta\psi_{aL}^*}$ and $i\frac{d}{dt}\psi_{aR} = \frac{\delta V_{\text{couple}}}{\delta\psi_{aR}^*}$, we can derive the equation of motion of the Josephson current in each phase and analyze the pseudo Goldstone modes. However, as we have mentioned above, the symmetry of the ground state in ferromagnetic phase only leads to an Abelian Josephson effect, which is not of interest in the present Letter. Therefore, we will just analyze the antiferromagnetic phase and cyclic phase which are important realizations of non-Abelian Josephson effect.

(I) Antiferromagnetic phase: To obtain the pseudo Goldstone modes, we need to linearize the equation of motion around the ground state $\psi_L = \psi_R = (\sqrt{n}/\sqrt{2})(1, 0, 0, 0, 1)$. By minimizing the potential V_{couple} under the above antiferromagnetic ground state, we get $n = (\mu + J)/(c_0 + c_2/5)$. With this ground state configu-
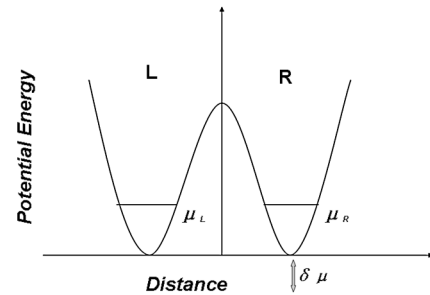

FIG. 1. The experimental schematic of a spin-2 Bose gas trapped in a double well with chemical potentials μ_L of left and μ_R of right trap, which initially satisfy $\mu_L = \mu_R$. To drive Josephson effect, we add a small distortion $\delta\mu$ to μ_R.

ration, we can obtain the linearized equations of the fluc-
tuations in this phase and analyze the excitation spectrum.

(i) The $m = 0$ mode. The equation of motion of the
$m = 0$ mode is given as

$$i\frac{d}{dt}\phi_{0L} = \left(-\frac{c_2}{5}n + J\right)\phi_{0L} + \frac{c_2}{5}n\phi_{0L}^* - J\phi_{0R}, \quad (3)$$

and a similar equation of ϕ_{0R}. Since this mode is
decoupled from others, it corresponds to an Abelian
Josephson current. By solving Eq. (3), we obtain the
eigenenergies of this mode, one is zero corresponding
to a massless Goldstone mode and the other is $\omega_0 =$
$2\sqrt{J(J + \frac{2|c_2|n}{5})}$ corresponding to a pseudo Goldstone
mode.

(ii) The $m = \pm 1$ coupled modes. The $m = \pm 1$ modes
are coupled in the following equations:

$$i\frac{d}{dt}\phi_{1L} = \left[n\left(c_1 - \frac{c_2}{5}\right) + J\right]\phi_{1L} + n\left(c_1 - \frac{c_2}{5}\right)\phi_{-1L}^*$$
$$- J\phi_{1R},$$
$$i\frac{d}{dt}\phi_{-1L} = \left[n\left(c_1 - \frac{c_2}{5}\right) + J\right]\phi_{-1L} + n\left(c_1 - \frac{c_2}{5}\right)\phi_{1L}^*$$
$$- J\phi_{-1R}, \quad (4)$$

and a similar set of equations of $\phi_{\pm 1R}$. The solution
involves two pseudo Goldstone modes with the same gap
of $\omega_{\pm 1} = 2\sqrt{n(c_1 - \frac{c_2}{5})J + J^2}$.

(iii) The $m = \pm 2$ coupled modes. The $m = \pm 2$
modes are coupled together in a set of equations similar
to the ones of the $m = \pm 1$ case. We will not list the
detailed equations of motion but just give the result
here and after. We obtain two pseudo Goldstone modes
with energy gap $\omega_{\pm 2}^{(1)} = 2\sqrt{n(c_0 + \frac{c_2}{5})J + J^2}$ and $\omega_{\pm 2}^{(2)} =$
$2\sqrt{n(c_1 - \frac{c_2}{20})J + J^2}$. We can see that there are a total of
five pseudo Goldstone modes with four different gaps in
this phase which consists with our analysis of the ground
state degeneracy. In Fig. 2 we give the dependence of the
above frequencies on coupling parameter J. Recently, a
polar behavior has been observed in the $F = 2$ ground state
of ^{87}Rb condensate [23]. We expect that the above modes
of fluctuations can be observed in this system in future
experiments. In the case of the ^{87}Rb system, the value of
interacting strengths under typical experimental conditions
are given as [23]: $c_1 n$: 0–10 nK, $c_2 n$: 0–0.2 nK, and $c_0 n$
about 150 nK. According to the weak coupling limit, we
assume that the coupling parameter J is much smaller than
the interaction energy of the condensate and given as about
0.1 nK. Under these conditions, we can obtain the frequen-
cies of the fluctuation related to the antiferromagnetic
phase, which is of order 100 Hz. The measurement of
fluctuations on this characteristic time scale (about
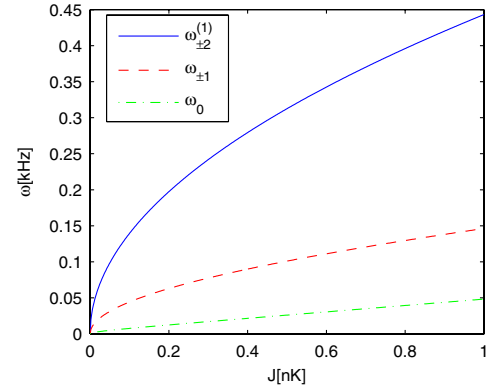10 ms) is accessible in current experiments.

FIG. 2 (color online).   The frequencies of the pseudo Goldstone
modes as a function of coupling parameter J in the case of
antiferromagnetic phase. All of the frequencies are proportional
to $\sqrt{J}$ when J approaches zero and the dependence on J becomes
linearizing when J is large compared to the interaction energy.

(II) Cyclic phase: Following the same procedure in
the antiferromagnetic phase we can obtain the
pseudo Goldstone modes for the cyclic phase.

(i) The $m = \pm 2, 0$ coupled modes. We find that each
Goldstone mode in the corresponding uncoupled system
[20] breaks into one massless mode and one
pseudo Goldstone mode. Since there are two Goldstone
modes in the uncoupled system, we find two
pseudo Goldstone modes with energy $\omega_{0,\pm 2}^{(1)} =$
$2\sqrt{J^2 + c_0 nJ}$ and $\omega_{0,\pm 2}^{(2)} = 2\sqrt{J^2 + 2c_1 nJ}$.

(ii) The $m = \pm 1$ coupled modes. As we know, there are
two massless Goldstone modes with the same energy in
the uncoupled system [20]. By solving the equation of
motions for this mode, we find that each of them leads
to a pseudo Goldstone mode with a gap $\omega_{\pm 1} =$
$2\sqrt{J^2 + 2c_1 nJ}$. There are four pseudo Goldstone modes
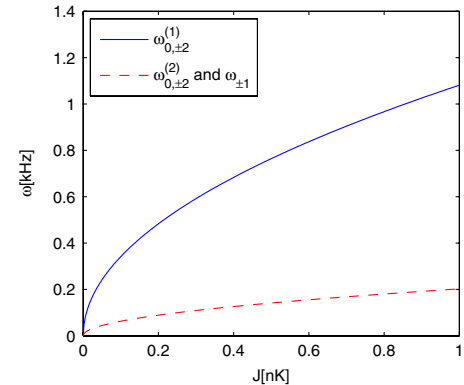in the cyclic phase which is consistent with our previous

FIG. 3 (color online).   The frequencies of pseudo Goldstone
modes as a function of coupling parameter J in the case of cyclic
phase. The dependence on J is similar to the case in antiferro-
magnetic phase. However, there are two degenerate modes $\omega_{0,\pm 2}^{(2)}$
and $\omega_{\pm 1}$ in this case, which are originated from the two degen-
erate Bogoliubov modes in the uncoupled system.

analysis on the symmetry of this phase. These kinds of fluctuations in cyclic phase are expected to be realized in a condensate of ^{85}Rb atoms [21]. Based on the current estimates of scattering lengths, the value of interacting strengths under typical experimental condition is given as: c_1n: 0–20 nK, c_2n: 0–0.6 nK, and c_0n about 600 nK. Under these conditions, we can also estimate the dc Josephson frequencies in cyclic phase, which is about 100–300 Hz. The dependence of the above frequencies on coupling parameter J is shown in Fig. 3.

Experimental signatures of Non-Abelian Josephson effect.—The experimental setup of a spin-2 Bose gas trapped in a double well is illustrated in Fig. 1. The dc non-Abelian Josephson current can be detected with the following steps. The first step is to initiate a density oscillation in the system. This can be realized by slightly changing the depth of one well, which will cause a small imbalance in chemical potential ($\mu_R \rightarrow \mu_R + \delta\mu$) between the two wells, and then tuning it back. The next step is to detect the time dependence of the particle numbers in different spin component. Such kind of detection can be realized by first spatially separating different spin components with a Stern-Gerlach method during time of flight after switching off the trapping potential. Then, the number of atoms in each spin component will be related to the respective spatial density distributions which can be evaluated by the absorption imaging method. Following the above steps, one can measure the density oscillation in each of the spin components which are coupled together due to the non-Abelian symmetry of the system. The measurement on the dependence of the oscillation frequencies on J can be realized by varying the barrier between the two wells and repeating the above measurement.

In mean field theory, the condensates of ^{87}Rb atom in $F = 2$ state are predicted to be polar ($c_1 - c_2/20 > 0$ and $c_2 < 0$), but close to the border to cyclic phase ($c_1 > 0$ and $c_2 > 0$) [21]. Furthermore, polar behavior in the $F = 2$ ground state of ^{87}Rb condensate has been observed in recent experiment [23]. As a result, we expect that the pseudo Goldstone modes of the antiferromagnetic phase could be observed in experiments. As we have mentioned, the value of interacting strengths under typical experimental conditions are given as [23]: c_1n: 0–10 nK, c_2n: 0–0.2 nK, and c_0n about 150 nK, which leads to the fluctuation time scale of about 10 ms in this system. On this time scale, the measurement we proposed above is completely accessible within recent experimental technique in $F = 2$ spinor Bose-Einstein condensates of ^{87}Rb system [23,24]. In order to observe this dynamical oscillation clearly in experiment, the temperature should be lower than the gap of the pseudo Goldstone modes, which is about 1–10 nk. Although there is still no such kind of measurement performed in a system with cyclic phase, we expect that it will be realized in a condensate of ^{85}Rb atoms in the near future [21].

In summary, we reveal a novel Josephson effect in spin-2 Bose system which involves non-Abelian symmetry and propose an experimental protocol to realize the so-called non-Abelian Josephson effect in this system. We find that the frequencies of pseudo Goldstone modes do not only relate to the coupling parameter but also to the interacting strengths, which is a nonlinear effect due to the spin dependent interaction. Our results are of particular significance for exploring the new features of the non-Abelian Josephson effect which are very distinct from the Abelian case.

[1] B. D. Josephson, Phys. Lett. **1**, 251 (1962).
[2] M. Albiez *et al.*, Phys. Rev. Lett. **95**, 010402 (2005).
[3] Y. Shin *et al.*, Phys. Rev. Lett. **95**, 170402 (2005).
[4] A. Smerzi, S. Fantoni, S. Giovanazzi, and S. R. Shenoy, Phys. Rev. Lett. **79**, 4950 (1997).
[5] A. Barone and G. Paterno, *Physics and Applications of the Josephson Effect* (Wiley, New York, 1982).
[6] M. S. Chang, Q. Qin, W. Zhang, L. You, and M. S. Chapman, Nature Phys. **1**, 111 (2005).
[7] F. P. Esposito, L. P. Guay, R. B. MacKenzie, M. B. Paranjape, and L. C. R. Wijewardhana, Phys. Rev. Lett. **98**, 241602 (2007).
[8] J. Davis and R. Packard, Rev. Mod. Phys. **74**, 741 (2002).
[9] K. Rajagopal and F. Wilczek, in *At the Frontier of Particle Physics: Handbook of QCD: Boris Ioffe Festschrift*, edited by M. Shifman (World Scientific, Singapore, 2001), Chap. 35.
[10] J. Ruseckas, G. Juzeliunas, P. Ohberg, and M. Fleischhauer, Phys. Rev. Lett. **95**, 010404 (2005).
[11] D. M. Stamper-Kurn *et al.*, Phys. Rev. Lett. **80**, 2027 (1998).
[12] T. Ohmi and K. Machida, J. Phys. Soc. Jpn. **67**, 1822 (1998).
[13] T.-L. Ho, Phys. Rev. Lett. **81**, 742 (1998).
[14] S. Ashhab and C. Lobo, Phys. Rev. A **66**, 013609 (2002).
[15] H. T. Ng, C. K. Law, and P. T. Leung, Phys. Rev. A **68**, 013604 (2003).
[16] A. J. Leggett, Rev. Mod. Phys. **73**, 307 (2001).
[17] O. E. Mustecaplioglu, M. Zhang, and L. You, Phys. Rev. A **71**, 053616 (2005).
[18] O. E. Mustecaplioglu, W. Zhang, and L. You, Phys. Rev. A **75**, 023605 (2007).
[19] R. Barnett, A. Turner, and E. Demler, Phys. Rev. Lett. **97**, 180412 (2006).
[20] M. Ueda and M. Koashi, Phys. Rev. A **65**, 063602 (2002).
[21] C. V. Ciobanu, S.-K. Yip, and T.-L. Ho, Phys. Rev. A **61**, 033607 (2000).
[22] R. Barnett, S. Mukerjee, and J. E. Moore, Phys. Rev. Lett. **100**, 240405 (2008).
[23] H. Schmaljohann *et al.*, Phys. Rev. Lett. **92**, 040402 (2004).
[24] M. S. Chang *et al.*, Phys. Rev. Lett. **92**, 140403 (2004).